

Realization of $U_q(\mathfrak{sp}_{2n})$ within the Differential Algebra on Quantum Symplectic Space

Naihong Hu* and Jiao Zhang*

Dedicated to Professor Christian Kassel in honor of his 65th birthday

ABSTRACT. We realize the Hopf algebra $U_q(\mathfrak{sp}_{2n})$ as an algebra of quantum differential operators on the quantum symplectic space $\mathcal{X}(f_s; \mathbb{R})$ and prove that $\mathcal{X}(f_s; \mathbb{R})$ is a $U_q(\mathfrak{sp}_{2n})$ -module algebra whose irreducible summands are just its homogeneous subspaces. We give a coherence realization for all the positive root vectors under the actions of Lusztig's braid automorphisms of $U_q(\mathfrak{sp}_{2n})$.

1. Introduction

Quantum analogues of differential forms and differential operators on quantum groups or Hopf algebras or quantum space have lots of studies since the end of 1980s ([26], [25], [12], [2], [8], [21], etc and references therein). Of influence is due to Woronowicz. As a main theme of noncommutative (differential) geometry, the general theory of bicovariant differential calculus on quantum groups or Hopf algebras as an approach to quantizing Lie groups has been developed in [26]. Woronowicz's axiomatic framework on bicovariant bimodules is used to construct the first order differential calculi (FODC) on Hopf algebras which coincide with the notion of Hopf bimodules (the defining condition of Yetter-Drinfeld modules also appeared in [26], as pointed out by [24]), as well as provides the Woronowicz's braiding. The coupled pairs of a quantum group and of the corresponding quantum space on which it coacts were intimately interrelated [23]. The covariant differential calculus on the quantum space \mathbb{C}_q^n was built by Wess-Zumino [25] so as to extend the covariant coaction of the quantum group $GL_q(n)$ to quantum derivatives.

Recall that for any bialgebra \mathcal{A} , by a quantum space for \mathcal{A} we mean a right \mathcal{A} -comodule algebra. Here, we take \mathcal{A} as a certain Hopf quotient of the FRT bialgebra $\mathcal{A}(R)$, which is related with a standard R -matrix R of the $ABCD$ series (cf. [12], [23]), and we set $\mathcal{X} := \mathcal{X}_r(f_s; \mathbb{R})$ (adopting the notation as in the book [12]), for the definitions of polynomials f_s for types $ABCD$, refer to Definitions 4, 8, 12 in Sections 9.2 & 9.3 [12]. Roughly speaking, when $U_q(\mathfrak{g})$ viewed as the Hopf

Key words and phrases. Quantum symplectic groups, quantum symplectic space, quantum differential operators, differential calculus, module algebra.

*N.H., supported by the NSFC (Grant No. 11271131).

*J.Z., correspondence author, supported by the NSFC (Grant Nos. 11101258, 11371238).

dual object of quantum group G_q for types $ABCD$, the mentioned quantum space \mathcal{X} is a left $U_q(\mathfrak{g})$ -module algebra. A benefit of this turning point of view allows us to extend the quantum enveloping algebra $U_q(\mathfrak{g})$ into a quantum enveloping parabolic subalgebra in a larger one via the crossed product construction. This actually constitutes a supporting evidence for the Majid's conjecture [18] on the rank-inductive construction of $U_q(\mathfrak{g})$'s via his double-bosonization procedure (see also a recent work [6] for confirming the Majid's claim in the classical cases).

For types B , D , under the assumption that q is not a root of unity, Fiore [2] started with R -matrices associated to quantum groups $SO_q(N)$ to define some quantum differential operators on the quantum Euclidean spaces \mathbb{R}_q^N and realized $U_{q^{-1}}(\mathfrak{so}_N)$ within the differential algebra $\text{Diff}(\mathbb{R}_q^N)$ such that \mathbb{R}_q^N is a left $U_{q^{-1}}(\mathfrak{g})$ -module algebra for $\mathfrak{g} = \mathfrak{so}_N$, ($N = 2n+1$ or $2n$), and further developed the corresponding quantum Euclidean geometry in his subsequent works.

For type A , there appeared several special discussions in the rank 1 case, see [20], [11], [25], etc. While for arbitrary rank, different from the Fiore's discussion [2], the first author [7] introduced the notion of quantum divided power algebra $\mathcal{A}_q(n)$ defined both for generic q and root of unity case, and defined q -derivatives over it and realized the U -module algebra structure of $\mathcal{A}_q(n)$ for $U = U_q(\mathfrak{sl}_n)$, $u_q(\mathfrak{sl}_n)$, and provided a coherence realization of all the positive root vectors in terms of the quantum differential operators we defined (in the so-called modified q -Weyl algebra $\mathcal{W}_q(2n)$) which are compatible with the actions of Lusztig's braid automorphisms ([17]). Especially, this work resulted in the definition of the quantum universal enveloping algebras of abelian Lie algebras for the first time, and even the new Hopf algebra structure so-called the n -rank Taft algebra (see [8], [13]) in root of unity case, as well as giving rise to some new development (see [4]) such as introducing new concepts, the quantum Grassmann algebra, quantum de Rham complexes and their cohomologies, together with the Loewy filtrations of a class of interesting indecomposable modules for Lusztig's small quantum group $u_q(\mathfrak{sl}_n)$.

For type C , it seems lack of the corresponding discussions over the quantum symplectic space in the literature. Here we consider the quantum enveloping algebra $U_q(\mathfrak{sp}_{2n})$ and its corresponding quantum symplectic space $\mathcal{X}(f_s; \mathbf{R})$. We assume that \mathbf{k} is a field of characteristic zero and q is not a root of unity. Here we define the q -analogues $\partial_i := \partial_q / \partial x_i$ of the classical partial derivatives and introduce left- and right- multiplications operators x_{i_L} and x_{i_R} as in [11]. Our discussion also doesn't use its R -matrix as in [2]. We consider the subalgebra U_q^{2n} generated by some quantum differential operators in the quantum differential algebra (the q -Weyl algebra of type C). Furthermore, we check the Serre relations of U_q^{2n} and show $\mathcal{X}(f_s; \mathbf{R})$ is a $U_q(\mathfrak{sp}_{2n})$ -module algebra. At last, we also show that the positive root vectors of $U_q(\mathfrak{sp}_{2n})$ defined by Lusztig's braid automorphisms in [17] can be specified precisely by the quantum differential operators we defined.

The paper is organized as follows. In Section 2, we recall the definition of the quantum symplectic space $\mathcal{X}(f_s; \mathbf{R})$, and derive some useful formulas. In Section 3, we define the quantum differential operators on $\mathcal{X}(f_s; \mathbf{R})$ and a subalgebra U_q^{2n} of $\text{Diff}(\mathcal{X}(f_s; \mathbf{R}))$. We prove that the generators of U_q^{2n} satisfy the Serre relations which makes U_q^{2n} a quotient algebra of $U_q(\mathfrak{sp}_{2n})$. We show that $\mathcal{X}(f_s; \mathbf{R})$ is a $U_q(\mathfrak{sp}_{2n})$ -module algebra whose irreducible summands are just its homogeneous subspaces. In Section 4, we provide inductive formulas to calculate all the positive root vectors under the actions of Lusztig's braid automorphisms of $U_q(\mathfrak{sp}_{2n})$ from

simple root vectors. In Section 5, we give a coherence realization for all the positive root vectors of $U_q(\mathfrak{sp}_{2n})$.

For simplicity we write \mathcal{X} for $\mathcal{X}(f_s; \mathbf{R})$. Let \mathbb{N}_0 (resp. \mathbb{N}) be the set of non-negative (resp. positive) integers respectively. Let \mathbf{k} be the underlying field of characteristic 0. Assume that q is invertible in \mathbf{k} and is not a root of unity.

2. Preliminaries

2.1. Recall that the q -number $[m]_q$ for $m \in \mathbb{Z}$ is defined by $[m]_q := \frac{q^m - q^{-m}}{q - q^{-1}}$. Note that $[0]_q = 0$. For $m \in \mathbb{N}$, the q -factorial is defined by setting $[m]_q! := [1]_q[2]_q \cdots [m]_q$ and $[0]_q! := 1$. The q -binomial coefficients are defined by

$$\begin{bmatrix} m \\ n \end{bmatrix}_q := \frac{[m]_q[m-1]_q \cdots [m-n+1]_q}{[1]_q[2]_q \cdots [n]_q}$$

for $m, n \in \mathbb{Z}$ with $n > 0$, and $\begin{bmatrix} m \\ 0 \end{bmatrix}_q := 1$. So if $n > m \geq 0$, then $\begin{bmatrix} m \\ n \end{bmatrix}_q = 0$. Set $[A, B]_v = AB - vBA$ for $v \in \mathbf{k}$. When $v = 1$, $[\cdot, \cdot]_1$ is the commutator $[\cdot, \cdot]$. The following three lemmas can be checked directly and will be used many times in Sections 4 and 5.

Lemma 2.1. For $u, v \in \mathbf{k}$ and $u \neq 0$, if $AB = uBA$, then

$$\begin{aligned} [A, BC]_v &= uB[A, C]_{v/u}, \\ [A, CB]_v &= [A, C]_{v/u}B, \\ [CA, B]_v &= u[C, B]_{v/u}A, \\ [AC, B]_v &= A[C, B]_{v/u}. \end{aligned} \tag{2.1}$$

Lemma 2.2. For $u, v, w \in \mathbf{k}$ and $u \neq 0$, if $AC = uCA$, then

$$\begin{aligned} [[A, B]_v, C]_w &= [A, [B, C]_{w/u}]_{uv}, \\ [[B, A]_v, C]_w &= u[[B, C]_{w/u}, A]_{v/u}. \end{aligned} \tag{2.2}$$

Lemma 2.3. We have

$$\begin{aligned} [A, B]_q &= -q[B, A]_{q^{-1}}, \\ [AB, C]_{q^2} &= A[B, C]_q + q[A, C]_q B, \\ [[A, B]_q, C]_q &= [A, [B, C]_{q^2}] + [[A, C]_q, B]_q. \end{aligned} \tag{2.3}$$

2.2. Recall that the simple roots of $\mathfrak{sp}_q(2n)$ are $\alpha_1 = 2\varepsilon_1$ and $\alpha_i = \varepsilon_i - \varepsilon_{i-1}$ for $2 \leq i \leq n$. Note that here α_1 is chosen to be longer than other simple roots. Let Δ^+ be the set of positive roots of \mathfrak{sp}_{2n} , then

$$\Delta^+ = \{2\varepsilon_i, \pm\varepsilon_l + \varepsilon_k, \mid 1 \leq i \leq n, 1 \leq l < k \leq n\}.$$

2.3. Recall that the quantum universal enveloping algebra $U_q(\mathfrak{sp}_{2n})$ generated by $\{E_i, F_i, K_i, K_i^{-1}, i \in I^+\}$ has the defining relations as follows:

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \tag{2.5}$$

$$K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j, \tag{2.6}$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{2.7}$$

$$(2.8) \quad \sum_{t=0}^{m_{ij}} (-1)^t \begin{bmatrix} m_{ij} \\ t \end{bmatrix}_{q_i} E_i^t E_j E_i^{m_{ij}-t} = 0, \quad i \neq j,$$

$$(2.9) \quad \sum_{t=0}^{m_{ij}} (-1)^t \begin{bmatrix} m_{ij} \\ t \end{bmatrix}_{q_i} F_i^t F_j F_i^{m_{ij}-t} = 0, \quad i \neq j,$$

where $q_i = q^{\frac{(\alpha_i, \alpha_i)}{2}}$, $a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$, $m_{ij} = 1 - a_{ij}$, and the Cartan matrix (a_{ij}) of \mathfrak{sp}_{2n} in our indices takes

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ -2 & 2 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Note that $q_1 = q^2$, $q_i = q$ for $1 < i \leq n$. The relations (2.8) and (2.9) are usually called the *Serre relations*.

The algebra $U_q(\mathfrak{sp}_{2n})$ is a Hopf algebra equipped with coproduct Δ , counit ε and antipode S defined by

$$(2.10) \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$

$$(2.11) \quad \Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1},$$

$$(2.12) \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i^{\pm 1}) = 1,$$

$$S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i, \quad S(K_i^{\pm 1}) = K_i^{\mp 1},$$

for $i \in I^+$.

2.4. Recall [12], the quantum symplectic space \mathcal{X} is called the coordinate algebra of the virtual quantum symplectic space of $\mathfrak{sp}_q(2n)$. Given $3 \leq n \in \mathbb{N}$, fix the index sets $I = \{-n, -n+1, \dots, -1, 1, \dots, n-1, n\}$ and $I^+ = \{1, \dots, n\}$. Set $\lambda = q - q^{-1}$. By Proposition 16 in Section 9.3.4 [12], the quantum space \mathcal{X} is the algebra with generators x_i , $i \in I$, and defining relations:

$$(2.13) \quad x_j x_i = q x_i x_j, \quad i < j, \quad -i \neq j,$$

$$(2.14) \quad x_i x_{-i} = q^2 x_{-i} x_i + q^2 \lambda \Omega_{i+1}, \quad i \in I^+,$$

where $\Omega_i := \sum_{i \leq j \leq n} q^{j-i} x_{-j} x_j$ for $i \in I^+$, and \mathcal{X} is a vector space with basis

$$\{x_{-n}^{\alpha_{-n}} \cdots x_n^{\alpha_n} \mid \alpha_{-n}, \dots, \alpha_n \in \mathbb{N}_0\}.$$

By definition, for $1 \leq i \leq n-1$, we have

$$(2.15) \quad \Omega_i = x_{-i} x_i + q \Omega_{i+1}.$$

From relations (2.13) and (2.14) we can obtain the following identities:

$$(2.16) \quad \Omega_i x_k = \begin{cases} q^2 x_k \Omega_i, & -n \leq k \leq -i, \\ x_k \Omega_i, & -i < k < i, \\ q^{-2} x_k \Omega_i, & i \leq k \leq n, \end{cases}$$

and

$$\Omega_i \Omega_j = \Omega_j \Omega_i, \quad i, j \in I^+.$$

Set $x^\alpha := x_{-n}^{\alpha_{-n}} \cdots x_n^{\alpha_n}$ and $\alpha := (\alpha_{-n}, \alpha_{1-n}, \dots, \alpha_n)$, where $\alpha_{-n}, \dots, \alpha_n \in \mathbb{N}_0$. We call the monomial $x_{i_1}^{\alpha_{i_1}} \cdots x_{i_m}^{\alpha_{i_m}}$ whose subscripts are placed in an increasing order a *normal monomial*. For convenience, we make the convention: $x_0 = 1$, $\alpha_0 = 0$. Write $\varepsilon_i = (0, \dots, 1, \dots, 0)$ with 1 in the i -position and 0 elsewhere. Then $\alpha = \sum_{i \in I} \alpha_i \varepsilon_i$. Set $|\alpha| = \sum_{i \in I} \alpha_i$. Thus $\mathcal{X} = \oplus_m \mathcal{X}^m$ is an \mathbb{N}_0 -graded algebra with $\mathcal{X}^m = \text{Span}_{\mathbf{k}}\{x^\alpha \mid |\alpha| = m\}$.

By induction using relations (2.13)–(2.16) we get

$$x_i x_{-i}^m = q^{2m} x_{-i}^m x_i + q^{m+1} \lambda[m]_q \Omega_{i+1} x_{-i}^{m-1}$$

and

$$x_i^m x_{-i} = q^{2m} x_{-i} x_i^m + q^{m+1} \lambda[m]_q \Omega_{i+1} x_i^{m-1}$$

for $i \in I^+$ and $m \in \mathbb{N}_0$. Hence, for $i \in I^+$, we have

$$(2.17) \quad x_{-i} x^\alpha = \left(\prod_{j=-n}^{-i-1} q^{\alpha_j} \right) x^{\alpha+\varepsilon_{-i}}, \quad x^\alpha x_i = \left(\prod_{j=i+1}^n q^{\alpha_j} \right) x^{\alpha+\varepsilon_i},$$

$$(2.18) \quad x_i x^\alpha = \left(\prod_{j=-n}^{i-1} q^{\alpha_j} \right) q^{\alpha_{-i}} x^{\alpha+\varepsilon_i} + \left(\prod_{j=-n}^{-i-1} q^{-\alpha_j} \right) q^{\alpha_{-i}+1} \lambda[\alpha_{-i}]_q \Omega_{i+1} x^{\alpha-\varepsilon_{-i}},$$

$$(2.19) \quad x^\alpha x_{-i} = \left(\prod_{j=1-i}^n q^{\alpha_j} \right) q^{\alpha_i} x^{\alpha+\varepsilon_{-i}} + \left(\prod_{k=i}^n q^{\alpha_k} \right) \left(\prod_{j=-n}^{-i-1} q^{-2\alpha_j} \right) q \lambda[\alpha_i]_q \Omega_{i+1} x^{\alpha-\varepsilon_i}.$$

The following lemma will be used later.

Lemma 2.4. *For $i \in I^+$ we have*

$$(2.20) \quad \Omega_i x^\alpha = \left(\prod_{l=-n}^{-i} q^{2\alpha_l} \right) \left(\sum_{j=i}^n q^{j-i+(\alpha_{1-j}+\dots+\alpha_{j-1})} x^{\alpha+\varepsilon_{-j}+\varepsilon_j} \right).$$

PROOF. We prove this lemma by induction on i from n to 1. From (2.13) and (2.16) we have

$$\begin{aligned} \Omega_n x^\alpha &= q^{2\alpha_{-n}} x_{-n}^{\alpha_{-n}} \Omega_n x_{1-n}^{\alpha_{1-n}} \cdots x_n^{\alpha_n} = q^{2\alpha_{-n}} x_{-n}^{\alpha_{-n}+1} x_n x_{1-n}^{\alpha_{1-n}} \cdots x_n^{\alpha_n} \\ &= q^{2\alpha_{-n}} q^{\alpha_{1-n}+\dots+\alpha_{n-1}} x^{\alpha+\varepsilon_{-n}+\varepsilon_n}. \end{aligned}$$

So the formula (2.20) holds for $i = n$. Suppose (2.20) holds for $i > 1$. Then from (2.13), (2.15) and (2.16) we obtain

$$\begin{aligned} \Omega_{i-1} x^\alpha &= \left(\prod_{l=-n}^{1-i} q^{2\alpha_l} \right) x_{-n}^{\alpha_{-n}} \cdots x_{1-i}^{\alpha_{1-i}} \Omega_{i-1} x_{2-i}^{\alpha_{2-i}} \cdots x_n^{\alpha_n} \\ &= \left(\prod_{l=-n}^{1-i} q^{2\alpha_l} \right) x_{-n}^{\alpha_{-n}} \cdots x_{1-i}^{\alpha_{1-i}} (x_{1-i} x_{i-1} + q \Omega_i) x_{2-i}^{\alpha_{2-i}} \cdots x_n^{\alpha_n} \\ &= \left(\prod_{l=-n}^{1-i} q^{2\alpha_l} \right) q^{\alpha_{2-i}+\dots+\alpha_{i-2}} x^{\alpha+\varepsilon_{1-i}+\varepsilon_{i-1}} + q^{1+2\alpha_{1-i}} \Omega_i x^\alpha. \end{aligned}$$

One concludes by applying the induction hypothesis. \square

3. Quantum Differential Operators on $\mathcal{X}(f_s; \mathbb{R})$

3.1. We define some quantum analogs of differential operators on \mathcal{X} .

Definition 3.1. For any normal monomial x^α and $i \in I$, set

$$\begin{aligned}\partial_i.x^\alpha &:= [\alpha_i]_q x^{\alpha - \varepsilon_i}, \\ x_{i_L}.x^\alpha &:= x_i x^\alpha, \\ x_{i_R}.x^\alpha &:= x^\alpha x_i, \\ \mu_i.x^\alpha &:= q^{\alpha_i} x^\alpha, \\ \mu_i^{-1}.x^\alpha &:= q^{-\alpha_i} x^\alpha.\end{aligned}$$

Let $\text{Diff}(\mathcal{X})$ be the unital algebra of differential operators on \mathcal{X} generated by ∂_i , x_{i_L} , x_{i_R} , μ_i and μ_i^{-1} with $i \in I$.

Since $\mu_k \mu_l = \mu_k \mu_l$, we write

$$\tau_i := \prod_{j=i}^n \mu_j \quad \text{and} \quad \tau_{-i} := \prod_{j=-n}^{-i} \mu_j$$

for $i \in I^+$. Now we define a subalgebra of $\text{Diff}(\mathcal{X})$.

Definition 3.2. For $i \in I^+$ with $i \geq 2$, set

$$\begin{aligned}e_1 &:= [2]_q^{-1} q^{-1} \mu_1^{-1} (\tau_{-2}^{-1} x_{-1_L} + q^2 \tau_2^{-1} x_{-1_R}) \partial_1, \\ f_1 &:= [2]_q^{-1} q^{-1} \mu_{-1}^{-1} (\tau_2^{-1} x_{1_R} + q^2 \tau_{-2}^{-1} x_{1_L}) \partial_{-1}, \\ k_1 &:= \mu_{-1}^2 \mu_1^{-2}, \\ e_i &:= \mu_{i-1} \mu_i^{-1} \tau_{-i-1}^{-1} x_{-i_L} \partial_{1-i} - \tau_i^{-1} x_{i-1_R} \partial_i, \\ f_i &:= -\mu_{1-i} \mu_{-i}^{-1} \tau_{i+1}^{-1} x_{i_R} \partial_{i-1} + \tau_{-i}^{-1} x_{1-i_L} \partial_{-i}, \\ k_i &:= \mu_{-i} \mu_{1-i}^{-1} \mu_{i-1} \mu_i^{-1}.\end{aligned}$$

Let U_q^{2n} be the subalgebra of $\text{Diff}(\mathcal{X})$ generated by $\{e_i, f_i, k_i, k_i^{-1} \mid i \in I^+\}$.

Applying the operators defined in Definition 3.2 to any normal monomial x^α , using Definition 3.1 and (2.17)–(2.19), we get

$$(3.1) \quad e_1.x^\alpha = [\alpha_1]_{q^2} x^{\alpha + \varepsilon_{-1} - \varepsilon_1} + \lambda \begin{bmatrix} \alpha_1 \\ 2 \end{bmatrix}_q q^{2-2(\alpha_{-n} + \dots + \alpha_{-2})} \Omega_2 x^{\alpha - 2\varepsilon_1},$$

$$(3.2) \quad f_1.x^\alpha = [\alpha_{-1}]_{q^2} x^{\alpha - \varepsilon_{-1} + \varepsilon_1} + \lambda \begin{bmatrix} \alpha_{-1} \\ 2 \end{bmatrix}_q q^{2-2(\alpha_{-n} + \dots + \alpha_{-2})} \Omega_2 x^{\alpha - 2\varepsilon_{-1}},$$

$$(3.3) \quad k_1.x^\alpha = q^{2(\alpha_{-1} - \alpha_1)} x^\alpha,$$

$$(3.4) \quad e_i.x^\alpha = q^{\alpha_{i-1} - \alpha_i} [\alpha_{1-i}]_q x^{\alpha + \varepsilon_{-i} - \varepsilon_{1-i}} - [\alpha_i]_q x^{\alpha + \varepsilon_{i-1} - \varepsilon_i},$$

$$(3.5) \quad f_i.x^\alpha = [\alpha_{-i}]_q x^{\alpha - \varepsilon_{-i} + \varepsilon_{1-i}} - q^{\alpha_{1-i} - \alpha_{-i}} [\alpha_{i-1}]_q x^{\alpha - \varepsilon_{i-1} + \varepsilon_i},$$

$$(3.6) \quad k_i.x^\alpha = q^{\alpha_{-i} - \alpha_{1-i} + \alpha_{i-1} - \alpha_i} x^\alpha,$$

for $1 < i \leq n$.

The following two lemmas will be used later.

Lemma 3.3. *For $i, j \in I^+$, we have*

$$e_i.\Omega_j = \begin{cases} 0, & i \neq j, \\ -x_{-j}x_{j-1}, & i = j > 1, \\ x_{-1}^2, & i = j = 1, \end{cases} \quad \text{and} \quad f_i.\Omega_j = \begin{cases} 0, & i \neq j, \\ x_{1-j}x_j, & i = j > 1, \\ x_1^2, & i = j = 1. \end{cases}$$

PROOF. It follows immediately from (3.1) and (3.5). \square

Lemma 3.4. *For any two normal monomials x^α, x^β and $i \in I^+$ we have*

$$\begin{aligned} k_i^{\pm 1}.(x^\alpha x^\beta) &= (k_i^{\pm 1}.x^\alpha)(k_i^{\pm 1}.x^\beta), \\ e_i.(x^\alpha x^\beta) &= (e_i.x^\alpha)(k_i.x^\beta) + x^\alpha(e_i.x^\beta), \\ f_i.(x^\alpha x^\beta) &= (f_i.x^\alpha)x^\beta + (k_i^{-1}.x^\alpha)(f_i.x^\beta). \end{aligned}$$

PROOF. We prove this lemma by induction on $|\alpha|$. For $|\alpha| = 1$, write $x^\alpha = x_j$, $j \in I$. Then this lemma can be derived from the relations (2.17)–(2.18)–(2.20), (3.1)–(3.6) and Lemma 3.3 directly. We omit this straightforward and lengthy verification. Suppose that the lemma holds for any normal monomial x^α with $|\alpha| = m$. Let x^γ be a normal monomial with $|\gamma| = m + 1$. We can write $x^\gamma = x_j x^\alpha$, where $|\alpha| = m$ and j is the smallest index in (r_{-n}, \dots, r_n) such that $\gamma_j \neq 0$. Since $x^\alpha x^\beta$ can be written as a linear combination of normal monomials, by the induction hypothesis, we get

$$\begin{aligned} k_i.(x^\gamma x^\beta) &= k_i.(x_j x^\alpha x^\beta) = (k_i.x_j)(k_i.(x^\alpha x^\beta)) \\ &= (k_i.x_j)(k_i.x^\alpha)(k_i.x^\beta) = (k_i.x^\gamma)(k_i.x^\beta). \end{aligned}$$

Then

$$\begin{aligned} e_i.(x^\gamma x^\beta) &= e_i.(x_j x^\alpha x^\beta) = (e_i.x_j)(k_i.(x^\alpha x^\beta)) + x_j(e_i.(x^\alpha x^\beta)) \\ &= (e_i.x_j)(k_i.x^\alpha)(k_i.x^\beta) + x_j(e_i.x^\alpha)(k_i.x^\beta) + x_j x^\alpha(e_i.x^\beta) \\ &= (e_i.(x_j x^\alpha))(k_i.x^\beta) + x^\gamma(e_i.x^\beta) = (e_i.x^\gamma)(k_i.x^\beta) + x^\gamma(e_i.x^\beta). \end{aligned}$$

Other relations can be proved similarly. \square

The following lemma can be easily checked by definition.

Lemma 3.5. *For any $m \in \mathbb{Z}$ we have*

$$(3.7) \quad [m+1]_q = q[m]_q + q^{-m} = q^{-1}[m]_q + q^m,$$

$$(3.8) \quad \begin{bmatrix} m+1 \\ 2 \end{bmatrix}_q - \begin{bmatrix} m \\ 2 \end{bmatrix}_q = [m]_{q^2},$$

$$(3.9) \quad \begin{bmatrix} m+1 \\ 2 \end{bmatrix}_q - q^2 \begin{bmatrix} m \\ 2 \end{bmatrix}_q = q^{1-m}[m]_q.$$

Now we state one of our main theorems.

Theorem 3.6. *The generators e_i, f_i, k_i, k_i^{-1} , $i \in I^+$, of U_q^{2n} satisfy the relations (2.5)–(2.9) after replacing E_i, F_i, K_i, K_i^{-1} by e_i, f_i, k_i, k_i^{-1} respectively. Hence, there is a unique surjective algebra homomorphism $\Psi : U_q(\mathfrak{sp}_{2n}) \rightarrow U_q^{2n}$ mapping E_i, F_i, K_i, K_i^{-1} to e_i, f_i, k_i, k_i^{-1} respectively.*

PROOF. The relations (2.5) are clear. Using (3.1)–(3.6) the relations (2.6) can be easily checked. For (2.7), we only prove the case $i = j = 1$, the others can be checked similarly. For any normal monomial x^α , using (3.1)–(3.3) and Lemmas 3.3 and 3.4, we get

$$\begin{aligned}
e_1 f_1 x^\alpha &= [\alpha_{-1}]_{q^2} e_1 x^{\alpha - \varepsilon_{-1} + \varepsilon_1} + \lambda \begin{bmatrix} \alpha_{-1} \\ 2 \end{bmatrix}_q q^{2-2(\alpha_{-n} + \dots + \alpha_{-2})} e_1 (\Omega_2 x^{\alpha - 2\varepsilon_{-1}}) \\
&= [\alpha_{-1}]_{q^2} e_1 x^{\alpha - \varepsilon_{-1} + \varepsilon_1} + \lambda \begin{bmatrix} \alpha_{-1} \\ 2 \end{bmatrix}_q q^{2-2(\alpha_{-n} + \dots + \alpha_{-2})} \Omega_2 (e_1 x^{\alpha - 2\varepsilon_{-1}}) \\
&= [\alpha_{-1}]_{q^2} [\alpha_1 + 1]_{q^2} x^\alpha \\
&\quad + \lambda \left(\begin{bmatrix} \alpha_{-1} + 1 \\ 2 \end{bmatrix}_q [\alpha_{-1}]_{q^2} + \begin{bmatrix} \alpha_{-1} \\ 2 \end{bmatrix}_q [\alpha_1]_{q^2} \right) q^{2-2(\alpha_{-n} + \dots + \alpha_{-2})} \Omega_2 x^{\alpha - \varepsilon_{-1} - \varepsilon_1} \\
&\quad + \lambda^2 \begin{bmatrix} \alpha_{-1} \\ 2 \end{bmatrix}_q \begin{bmatrix} \alpha_1 \\ 2 \end{bmatrix}_q q^{4-4(\alpha_{-n} + \dots + \alpha_{-2})} \Omega_2 \Omega_2 x^{\alpha - 2\varepsilon_{-1} - 2\varepsilon_1}
\end{aligned}$$

and

$$\begin{aligned}
f_1 e_1 x^\alpha &= [\alpha_1]_{q^2} f_1 x^{\alpha + \varepsilon_{-1} - \varepsilon_1} + \lambda \begin{bmatrix} \alpha_1 \\ 2 \end{bmatrix}_q q^{2-2(\alpha_{-n} + \dots + \alpha_{-2})} f_1 (\Omega_2 x^{\alpha - 2\varepsilon_1}) \\
&= [\alpha_1]_{q^2} f_1 x^{\alpha + \varepsilon_{-1} - \varepsilon_1} + \lambda \begin{bmatrix} \alpha_1 \\ 2 \end{bmatrix}_q q^{2-2(\alpha_{-n} + \dots + \alpha_{-2})} (k_1^{-1} \cdot \Omega_2) (f_1 x^{\alpha - 2\varepsilon_1}) \\
&= [\alpha_1]_{q^2} [\alpha_{-1} + 1]_{q^2} x^\alpha \\
&\quad + \lambda \left(\begin{bmatrix} \alpha_{-1} + 1 \\ 2 \end{bmatrix}_q [\alpha_1]_{q^2} + \begin{bmatrix} \alpha_1 \\ 2 \end{bmatrix}_q [\alpha_{-1}]_{q^2} \right) q^{2-2(\alpha_{-n} + \dots + \alpha_{-2})} \Omega_2 x^{\alpha - \varepsilon_{-1} - \varepsilon_1} \\
&\quad + \lambda^2 \begin{bmatrix} \alpha_{-1} \\ 2 \end{bmatrix}_q \begin{bmatrix} \alpha_1 \\ 2 \end{bmatrix}_q q^{4-4(\alpha_{-n} + \dots + \alpha_{-2})} \Omega_2 \Omega_2 x^{\alpha - 2\varepsilon_{-1} - 2\varepsilon_1}.
\end{aligned}$$

Using (3.7) and (3.8) we obtain

$$\begin{aligned}
[e_1, f_1] x^\alpha &= ([\alpha_{-1}]_{q^2} [\alpha_1 + 1]_{q^2} - [\alpha_1]_{q^2} [\alpha_{-1} + 1]_{q^2}) x^\alpha \\
&= ([\alpha_{-1}]_{q^2} q^{2\alpha_1} - [\alpha_1]_{q^2} q^{2\alpha_{-1}}) x^\alpha = [\alpha_{-1} - \alpha_1]_{q^2} x^\alpha.
\end{aligned}$$

Since $q_1 = q^2$,

$$\frac{k_1 - k_1^{-1}}{q_1 - q_1^{-1}} x^\alpha = \frac{q^{2(\alpha_{-1} - \alpha_1)} - q^{-2(\alpha_{-1} - \alpha_1)}}{q^2 - q^{-2}} x^\alpha = [\alpha_{-1} - \alpha_1]_{q^2} x^\alpha.$$

Hence

$$[e_1, f_1] = \frac{k_1 - k_1^{-1}}{q_1 - q_1^{-1}}.$$

Consider the first Serre relation (2.8). For the case $i = 1$ and $j = 2$, we need to prove

$$e_1^2 e_2 - [2]_{q^2} e_1 e_2 e_1 + e_2 e_1^2 = 0.$$

Set $e_{1,2} := [e_1, e_2]_{q^2}$. It is equivalent to show

$$(3.10) \quad [e_1, e_{1,2}]_{q^{-2}} = 0.$$

By (3.1), (3.4), (3.6) and Lemmas 3.3–3.5, we get

$$\begin{aligned}
e_{1,2} x^\alpha &= -[\alpha_1]_q q^{2+\alpha_{-1}-\alpha_2} x^{\alpha + \varepsilon_{-2} - \varepsilon_1} - [\alpha_2]_q q^{-2\alpha_1} x^{\alpha + \varepsilon_{-1} - \varepsilon_2} \\
&\quad - \lambda [\alpha_1]_q [\alpha_2]_q q^{3-2(\alpha_{-n} + \dots + \alpha_{-2}) - \alpha_1} \Omega_2 x^{\alpha - \varepsilon_1 - \varepsilon_2}.
\end{aligned}$$

From Lemmas 3.3 and 3.4 and the relation $k_2 e_1 k_2^{-1} = q^{-2} e_1$ which has been proved before, it is easy to show

$$e_{1,2}(\Omega_2 x^\alpha) = \Omega_2(e_{1,2} x^\alpha) + (e_1 e_2 \Omega_2)(k_1 k_2 x^\alpha).$$

Using the above two formulas and the identity

$$q[\alpha_1]_q [\alpha_1 - 1]_{q^2} - [\alpha_1]_{q^2} [\alpha_1 - 1]_q - \lambda \begin{bmatrix} \alpha_1 \\ 2 \end{bmatrix}_q q^{1-\alpha_1} = 0$$

which is easy to check, we can verify $[e_1, e_{1,2}]_{q^{-2}} x^\alpha = 0$ by direct computation. So the relation (3.10) holds.

Consider the first Serre relation (2.8) for $i = 2, j = 1$. We need to prove

$$e_2^3 e_1 - [3]_q e_2^2 e_1 e_2 + [3]_q e_2 e_1 e_2^2 - e_1 e_2^3 = 0.$$

It is equivalent to show

$$(3.11) \quad [e_2, [e_{1,2}, e_2]]_{q^2} = 0.$$

In order to verify the first Serre relation (2.8) for $j = i \pm 1$ and $i, j > 1$, i.e.,

$$e_i^2 e_{i \pm 1} - [2]_q e_i e_{i \pm 1} e_i + e_{i \pm 1} e_i^2 = 0,$$

which is equivalent to

$$(3.12) \quad [e_i, [e_i, e_{i \pm 1}]]_{q^{-1}} = 0.$$

It is not hard to check (3.11) and (3.12) after applying their left hand sides to x^α . For $|i - j| > 2$ the first Serre relation is $[e_i, e_j] = 0$ which is obvious.

The second Serre relation can be verified similarly. This completes the proof. \square

Due to the above theorem, we can realize the elements of the quantum group $U_q(\mathfrak{sp}_{2n})$ as certain q -differential operators on \mathcal{X} , then \mathcal{X} is a left $U_q(\mathfrak{sp}_{2n})$ -module.

Let $(H, m, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra. Recall that an algebra A is called a left H -module algebra if A is a left H -module, and the multiplication map and the unit map of A are left H -module homomorphisms, that is,

$$(3.13) \quad h.1_A = \varepsilon(h)1_A,$$

$$(3.14) \quad h.(ab) = \sum (h_{(1)}.a)(h_{(2)}.b),$$

for any $h \in H$, $a, b \in A$, where $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$.

Theorem 3.7. *The algebra \mathcal{X} is a left $U_q(\mathfrak{sp}_{2n})$ -module algebra.*

PROOF. It is sufficient to check (3.13) and (3.14) on the generators of $U_q(\mathfrak{sp}_{2n})$, since ε and Δ are algebra homomorphisms. The relations (2.12) and (3.1)–(3.6) imply (3.13). Lemma 3.4, (2.10) and (2.11) imply (3.14). \square

3.2. We consider the decomposition of \mathcal{X} into a direct sum of irreducible $U_q(\mathfrak{sp}_{2n})$ -submodules. Recall that $\mathcal{X} = \bigoplus_{m \in \mathbb{N}_0} \mathcal{X}^m$, where \mathcal{X}^m is the subspace of homogeneous elements of degree m .

Proposition 3.8. *The vector space \mathcal{X}^m is a finite dimensional irreducible $U_q(\mathfrak{sp}_{2n})$ -module with highest weight vector x_n^m and highest weight $m\varepsilon_n$.*

PROOF. If $m = 0$, then $\mathcal{X}^m = \mathbf{k}$ is a one dimensional vector space. It is easy to see that \mathbf{k} is an irreducible $U_q(\mathfrak{sp}_{2n})$ -module.

Consider the case $m > 0$. It is clear that the vector space \mathcal{X}^m is a $U_q(\mathfrak{sp}_{2n})$ -submodule of \mathcal{X} , since E_i, F_i, K_i preserve the degrees of homogeneous elements. The \mathbf{k} -basis of \mathcal{X}^m is $\{x_{i_1}x_{i_2}\cdots x_{i_m} \mid i_1 \leq i_2 \leq \cdots \leq i_m, i_1, \dots, i_m \in I\}$, so

$$\dim \mathcal{X}^m = \binom{m+2n-1}{m}.$$

Consider the vector x_{-n}^m . It is easy to check that for any $i \in I^+$

$$E_i \cdot x_{-n}^m = 0 \quad \text{and} \quad K_i \cdot x_{-n}^m = q^{m\delta_{i,n}} x_{-n}^m.$$

Hence x_{-n}^m is a highest weight vector. Recall that according to our choice of index the simple roots of \mathfrak{sp}_{2n} are $\alpha_1 = 2\varepsilon_1$ and $\alpha_i = \varepsilon_i - \varepsilon_{i-1}$ for $1 < i \leq n$, and the set of positive roots of \mathfrak{sp}_{2n} is

$$\Delta^+ = \{2\varepsilon_i, \varepsilon_k \pm \varepsilon_l \mid 1 \leq i \leq n, 1 \leq l < k \leq n\},$$

then δ the half-sum of all positive roots of \mathfrak{sp}_{2n} is

$$\delta = \varepsilon_1 + 2\varepsilon_2 + \cdots + n\varepsilon_n.$$

Let $\lambda = m\varepsilon_n$. It yields that $(\lambda, \alpha_i) = m\delta_{i,n}$ for any $i \in I^+$, so λ is a dominant integral weight. Recall (see [10, 12]) that the theory of finite dimensional representations of $U_q(\mathfrak{g})$ is very similar to that of \mathfrak{g} when q is not a root of unity. The vector x_{-n}^m generates a finite dimensional irreducible $U_q(\mathfrak{sp}_{2n})$ -module of highest weight λ , denoted by $V(\lambda)$. So $V(\lambda)$ is a submodule of \mathcal{X}^m . From

$$(\delta, 2\varepsilon_i) = 2i, \quad (\delta, \varepsilon_k - \varepsilon_l) = k - l, \quad (\delta, \varepsilon_k + \varepsilon_l) = k + l,$$

and

$$(\lambda, 2\varepsilon_i) = 2m\delta_{i,n}, \quad (\lambda, \varepsilon_k - \varepsilon_l) = m\delta_{k,n}, \quad (\lambda, \varepsilon_k + \varepsilon_l) = m\delta_{k,n},$$

by a corollary of Weyl's formula ([9], 24.3), we get

$$\begin{aligned} \dim V(\lambda) &= \frac{\prod_{\alpha \in \Delta^+} (\lambda + \delta, \alpha)}{\prod_{\alpha \in \Delta^+} (\delta, \alpha)} = \frac{2(m+n) \prod_{l=1}^{n-1} (m+n-l)(m+n+l)}{2n \prod_{l=1}^{n-1} (n-l)(n+l)} \\ &= \frac{(m+2n-1)!}{(2n-1)!m!} = \binom{m+2n-1}{m}. \end{aligned}$$

Then $\dim V(\lambda) = \dim \mathcal{X}^m$, so $\mathcal{X}^m = V(\lambda)$ is a finite dimensional irreducible $U_q(\mathfrak{sp}_{2n})$ -module with highest weight vector x_{-n}^m and highest weight $m\varepsilon_n$. \square

4. Positive root vectors of $U_q(\mathfrak{sp}_{2n})$

We are going to list all positive root vectors of $U_q(\mathfrak{sp}_{2n})$ in U_q^{2n} . We first recall some notions.

4.1. Let $s_i, i \in I^+$, be the reflections determined by the simple roots α_i . The braid group \mathcal{B} associated with \mathfrak{sp}_{2n} is the group generated by elements s_1, \dots, s_n subject to the relations

$$s_i s_j s_i s_j \cdots = s_j s_i s_j s_i \cdots, \quad i \neq j,$$

where we have $1 - a_{ij}$ factors on each side. Lusztig introduced actions of braid groups on $U_q(\mathfrak{g})$ in [14]–[17]. The following two propositions can be found in many books, for example [17, 10, 12], etc.

Proposition 4.1. *To every $i, i \in I^+$, there corresponds an algebra automorphism T_i of $U_q(\mathfrak{sp}_{2n})$ which acts on the generators K_j, E_j, F_j as*

$$\begin{aligned} T_i(K_j) &= K_j K_i^{-a_{ij}}, \quad T_i(E_i) = -F_i K_i^{-1}, \quad T_i(F_i) = -K_i E_i, \\ T_i(E_j) &= \sum_{r=0}^{-a_{ij}} (-1)^r q_i^r E_i^{(-a_{ij}-r)} E_j E_i^{(r)}, \quad \text{for } i \neq j, \\ T_i(F_j) &= \sum_{r=0}^{-a_{ij}} (-1)^r q_i^{-r} F_i^{(r)} F_j F_i^{(-a_{ij}-r)}, \quad \text{for } i \neq j, \end{aligned}$$

where $E_i^{(r)} = E_i^r / [r]_q!$ and $F_i^{(r)} = F_i^r / [r]_q!$. The mapping $s_i \mapsto T_i$ determines a homomorphism of the braid group \mathcal{B} into the group of algebra automorphisms of $U_q(\mathfrak{sp}_{2n})$.

The operators T_i defined by Proposition 4.1 is Lusztig's $T''_{i,-1}$ ([17], 37.1.3).

Proposition 4.2. *The operators T_i satisfy the following relations.*

(1) *For $i, j \in I^+$ with $|i - j| > 1$ we have*

$$T_i(E_j) = E_j, \quad T_i T_j = T_j T_i.$$

(2) *For $2 \leq i, j \leq n$ with $|i - j| = 1$ we have*

$$T_i(E_j) = [E_i, E_j]_q, \quad T_i T_j(E_i) = E_j, \quad T_i T_j T_i = T_j T_i T_j.$$

(3) *For $1 \leq i \neq j \leq 2$ we have*

$$\begin{aligned} T_1(E_2) &= [E_1, E_2]_{q^2}, \quad [2]_q T_2(E_1) = [E_2, [E_2, E_1]_{q^2}], \\ T_1 T_2 T_1(E_2) &= E_2, \quad T_2 T_1 T_2(E_1) = E_1, \\ T_1 T_2 T_1 T_2 &= T_2 T_1 T_2 T_1. \end{aligned}$$

The Weyl group W of \mathfrak{sp}_{2n} generated by s_i has the longest element w_0 whose reduced expression is

$$w_0 = \gamma_1 \cdots \gamma_n,$$

where $\gamma_i = s_i s_{i-1} \cdots s_1 \cdots s_{i-1} s_i$ (cf. [1]). Write $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$ for this reduced expression. Then

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \dots, \quad \beta_N = s_{i_1} s_{i_2} \cdots s_{i_{N-1}}(\alpha_{i_N})$$

exhaust all positive roots of \mathfrak{sp}_{2n} .

Definition 4.3. The elements

$$E_{\beta_r} = T_{i_1} T_{i_2} \cdots T_{i_{r-1}}(E_{i_r})$$

are called *positive root vectors* of $U_q(\mathfrak{sp}_{2n})$ corresponding to the roots β_r .

Set

$$\alpha_{i,i} = 2\varepsilon_i \text{ and } \alpha_{\pm l,k} = \pm \varepsilon_l + \varepsilon_k$$

for $1 \leq i \leq n$ and $1 \leq l < k \leq n$. We can list all positive roots in the ordering according to the longest element w_0 as follows:

$$\begin{aligned} &\alpha_{1,1}, \\ &\alpha_{1,2}, \alpha_{2,2}, \alpha_{-1,2}, \\ &\alpha_{2,3}, \alpha_{1,3}, \alpha_{3,3}, \alpha_{-1,3}, \alpha_{-2,3}, \\ &\dots, \end{aligned}$$

$$\alpha_{n-1,n}, \alpha_{n-2,n}, \dots, \alpha_{1,n}, \alpha_{n,n}, \alpha_{-1,n}, \alpha_{-2,n}, \dots, \alpha_{1-n,n}.$$

Write

$$E_{i,i} = E_{\alpha_{i,i}} \quad \text{and} \quad E_{\pm l,k} = E_{\alpha_{\pm l,k}}.$$

It is clear that $E_{1,1} = E_1$ and we will check in Corollary 4.5 that $E_{\alpha_i} = E_i$ for all $1 < i \leq n$. Set

$$T_{\gamma_i} = T_i T_{i-1} \cdots T_1 \cdots T_{i-1} T_i.$$

By Definition 4.3, all the positive root vectors of $U_q(\mathfrak{sp}_{2n})$ associated to the above ordering of Δ^+ is as follows, for any $1 < j \leq n$,

$$(4.1) \quad E_{j-1,j} = T_{\gamma_1} \cdots T_{\gamma_{j-1}}(E_j),$$

$$(4.2) \quad E_{i,j} = T_{\gamma_1} \cdots T_{\gamma_{j-1}} T_j T_{j-1} \cdots T_{i+2}(E_{i+1}), \quad \text{for } 1 \leq i < j-1,$$

$$(4.3) \quad E_{j,j} = T_{\gamma_1} \cdots T_{\gamma_{j-1}} T_j T_{j-1} \cdots T_2(E_1),$$

$$(4.4) \quad E_{-i,j} = T_{\gamma_1} \cdots T_{\gamma_{j-1}} T_j \cdots T_1 \cdots T_i(E_{i+1}), \quad \text{for } 1 \leq i < j.$$

Lemma 4.4. *For $1 < i \leq j \leq n$, we have*

$$(4.5) \quad T_1([E_2, E_1]_{q^2}) = E_2,$$

$$(4.6) \quad T_{\gamma_i} T_{i+1}(E_i) = [E_i, E_{i+1}]_q,$$

$$(4.7) \quad T_{\gamma_{j-1}} T_j T_{\gamma_{j-1}}(E_j) = E_j,$$

$$(4.8) \quad T_{\gamma_j} T_{j+1} T_j \cdots T_i = (T_j T_{j+1})(T_{j-1} T_j) \cdots (T_i T_{i+1}) T_{\gamma_{i-1}}(T_i T_{i+1} \cdots T_{j+1}).$$

PROOF. It is easy to see from (2.6) and (2.7) that

$$[[E_1, E_2]_{q^2}, K_1^{-1}]_{q^2} = 0 \quad \text{and} \quad [E_2, F_1] = 0,$$

then by Proposition 4.1 and relations (2.1)–(2.2) we get

$$\begin{aligned} T_1([E_2, E_1]_{q^2}) &= [T_1(E_2), T_1(E_1)]_{q^2} = [[E_1, E_2]_{q^2}, -F_1 K_1^{-1}]_{q^2} \\ &= -[[E_1, E_2]_{q^2}, F_1] K_1^{-1} = -[[E_1, F_1], E_2]_{q^2} K_1^{-1} \\ &= -\left[\frac{K_1 - K_1^{-1}}{q^2 - q^{-2}}, E_2\right]_{q^2} K_1^{-1} = E_2. \end{aligned}$$

The relation (4.6) is clear, since for $i > 1$ we have

$$T_{\gamma_i} T_{i+1}(E_i) = T_i T_{\gamma_{i-1}} T_i T_{i+1}(E_i) = T_i T_{\gamma_{i-1}}(E_{i+1}) = T_i(E_{i+1}) = [E_i, E_{i+1}]_q.$$

We use induction on j to prove (4.7). For $j = 2$ this is obvious by Proposition 4.2 (III). Now suppose that (4.7) holds for some j with $2 < j < n$. Then Proposition 4.2 (II) and induction yield

$$\begin{aligned} T_{\gamma_j} T_{j+1} T_j(E_{j+1}) &= T_j T_{\gamma_{j-1}}(T_j T_{j+1} T_j) T_{\gamma_{j-1}} T_j(E_{j+1}) \\ &= T_j T_{\gamma_{j-1}}(T_{j+1} T_j T_{j+1}) T_{\gamma_{j-1}} T_j(E_{j+1}) \\ &= T_j T_{j+1} T_{\gamma_{j-1}} T_j T_{\gamma_{j-1}}(T_{j+1} T_j(E_{j+1})) \\ &= T_j T_{j+1}(T_{\gamma_{j-1}} T_j T_{\gamma_{j-1}}(E_j)) \\ &= T_j T_{j+1}(E_j) = E_{j+1}. \end{aligned}$$

To prove (4.8) we use induction on $j - i$. For $j - i = 0$ we have

$$T_{\gamma_j} T_{j+1} T_j = T_j T_{\gamma_{j-1}} T_j T_{j+1} T_j = T_j T_{\gamma_{j-1}} T_{j+1} T_j T_{j+1} = T_j T_{j+1} T_{\gamma_{j-1}} T_j T_{j+1}.$$

Suppose that (4.8) holds for some $j - i - 1 > 0$. Then by induction we get

$$T_{\gamma_j} T_{j+1} T_j \cdots T_{i+1} T_i = (T_j T_{j+1})(T_{j-1} T_j) \cdots (T_{i+1} T_{i+2}) T_{\gamma_i}(T_{i+1} T_{i+2} \cdots T_{j+1}) T_i$$

$$\begin{aligned}
&= (T_j T_{j+1})(T_{j-1} T_j) \cdots (T_{i+1} T_{i+2})(T_{\gamma_i} T_{i+1} T_i) T_{i+2} \cdots T_{j+1} \\
&= (T_j T_{j+1})(T_{j-1} T_j) \cdots (T_{i+1} T_{i+2})(T_i T_{i+1}) T_{\gamma_{i-1}} (T_i T_{i+1} T_{i+2} \cdots T_{j+1}).
\end{aligned}$$

So (4.8) holds. \square

Using (4.7) and Proposition 4.2 (I) we get the following corollary easily.

Corollary 4.5. For any $1 < j \leq n$ we have

$$T_{\gamma_1} \cdots T_{\gamma_{j-1}} T_j \cdots T_1 \cdots T_{j-1} (E_j) = E_j,$$

that is, $E_{1-j,j} = E_{\alpha_j} = E_j$.

Proposition 4.6. The positive root vectors of $U_q(\mathfrak{sp}_{2n})$ have the following commutation relations:

$$(4.9) \quad E_{1,2} = [E_1, E_2]_q,$$

$$(4.10) \quad E_{-i,j} = [E_{-i,j-1}, E_j]_q, \quad 3 \leq i+2 \leq j \leq n,$$

$$(4.11) \quad E_{i,j} = [E_{i,j-1}, E_j]_q, \quad 3 \leq i+2 \leq j \leq n,$$

$$(4.12) \quad E_{j-1,j} = [E_{j-1}, E_{j-2,j}]_q, \quad 3 \leq j \leq n,$$

$$(4.13) \quad E_{j,j} = [2]_q^{-1} [E_{1,j}, E_{-1,j}], \quad 2 \leq j \leq n.$$

PROOF. Relation (4.9) is clear. For $i \geq 1$ by Proposition 4.2 and the identity (4.8) we obtain that

$$\begin{aligned}
&T_{\gamma_{i+1}} T_{i+2} T_{i+1} T_{\gamma_i} (E_{i+1}) \\
&= T_{i+1} T_{i+2} T_{\gamma_i} T_{i+1} T_{i+2} T_{\gamma_i} (E_{i+1}) = T_{i+1} T_{i+2} T_{\gamma_i} T_{i+1} T_{\gamma_i} ([E_{i+2}, E_{i+1}]_q) \\
&= [T_{i+1} T_{\gamma_i} T_{i+2} T_{i+1} (E_{i+2}), T_{i+1} T_{i+2} T_{\gamma_i} T_{i+1} T_{\gamma_i} (E_{i+1})]_q \\
&= [T_{i+1} T_{\gamma_i} T_{i+2} T_{i+1} (E_{i+2}), T_{i+1} T_{i+2} (E_{i+1})]_q \\
&= [T_{i+1} T_{\gamma_i} (E_{i+1}), E_{i+2}]_q.
\end{aligned}$$

So Proposition 4.2, (4.4), (4.8) and the above formula show that for $1 \leq i < j-1$

$$\begin{aligned}
E_{-i,j} &= T_{\gamma_1} \cdots T_{\gamma_{j-1}} T_j \cdots T_1 \cdots T_i (E_{i+1}) \\
&= (T_{\gamma_1} \cdots T_{\gamma_{j-2}}) (T_{\gamma_{j-1}} T_j T_{j-1} \cdots T_{i+1}) T_{\gamma_i} (E_{i+1}) \\
&= (T_{\gamma_1} \cdots T_{\gamma_{j-2}}) (T_{j-1} T_j) \cdots (T_{i+2} T_{i+3}) (T_{i+1} T_{i+2}) T_{\gamma_i} (T_{i+1} \cdots T_{j-1} T_j) T_{\gamma_i} (E_{i+1}) \\
&= (T_{\gamma_1} \cdots T_{\gamma_{j-2}}) (T_{j-1} T_j) \cdots (T_{i+2} T_{i+3}) T_{i+1} T_{\gamma_i} T_{i+2} T_{i+1} T_{i+2} T_{\gamma_i} (E_{i+1}) \\
&= (T_{\gamma_1} \cdots T_{\gamma_{j-2}}) (T_{j-1} T_j) \cdots (T_{i+2} T_{i+3}) T_{i+1} T_{\gamma_i} T_{i+1} T_{i+2} T_{i+1} T_{\gamma_i} (E_{i+1}) \\
&= (T_{\gamma_1} \cdots T_{\gamma_{j-2}}) (T_{j-1} T_j) \cdots (T_{i+2} T_{i+3}) T_{\gamma_{i+1}} T_{i+2} T_{i+1} T_{\gamma_i} (E_{i+1}) \\
&= (T_{\gamma_1} \cdots T_{\gamma_{j-2}}) (T_{j-1} T_j) \cdots (T_{i+2} T_{i+3}) [T_{i+1} T_{\gamma_i} (E_{i+1}), E_{i+2}]_q \\
&= (T_{\gamma_1} \cdots T_{\gamma_{j-2}}) [T_{j-1} \cdots T_{i+2} T_{i+1} T_{\gamma_i} (E_{i+1}), (T_{j-1} T_j) \cdots (T_{i+2} T_{i+3}) (E_{i+2})]_q \\
&= [T_{\gamma_1} \cdots T_{\gamma_{j-2}} T_{j-1} \cdots T_{i+2} T_{i+1} T_{\gamma_i} (E_{i+1}), E_j]_q \\
&= [E_{-i,j-1}, E_j]_q.
\end{aligned}$$

Hence the relation (4.10) holds.

For $j = i+2$ the relations (4.2) and (4.6) yield

$$E_{i,i+2} = T_{\gamma_1} \cdots T_{\gamma_{i+1}} T_{i+2} (E_{i+1}) = T_{\gamma_1} \cdots T_{\gamma_i} ([E_{i+1}, E_{i+2}]_q) = [E_{i,i+1}, E_{i+2}]_q.$$

For $j > i+2$ by (4.2), (4.6) and (4.8) we have

$$E_{i,j} = T_{\gamma_1} \cdots T_{\gamma_{j-1}} T_j T_{j-1} \cdots T_{i+2} (E_{i+1})$$

$$\begin{aligned}
&= T_{\gamma_1} \cdots T_{\gamma_{j-2}}(T_{j-1}T_j)(T_{j-2}T_{j-1}) \cdots (T_{i+2}T_{i+3})T_{\gamma_{i+1}}T_{i+2} \cdots T_{j-1}T_j(E_{i+1}) \\
&= T_{\gamma_1} \cdots T_{\gamma_{j-2}}(T_{j-1}T_j)(T_{j-2}T_{j-1}) \cdots (T_{i+2}T_{i+3})T_{\gamma_{i+1}}T_{i+2}(E_{i+1}) \\
&= T_{\gamma_1} \cdots T_{\gamma_{j-2}}(T_{j-1}T_j)(T_{j-2}T_{j-1}) \cdots (T_{i+2}T_{i+3})([E_{i+1}, E_{i+2}]_q) \\
&= [T_{\gamma_1} \cdots T_{\gamma_{j-2}}T_{j-1}T_{j-2} \cdots T_{i+2}(E_{i+1}), E_j]_q = [E_{i,j-1}, E_j]_q.
\end{aligned}$$

So the relation (4.11) holds.

For $j \geq 3$, using the relations (4.1) (4.4) and (4.11), we have

$$\begin{aligned}
E_{j-1,j} &= T_{\gamma_1} \cdots T_{\gamma_{j-1}}(E_j) = T_{\gamma_1} \cdots T_{\gamma_{j-2}}T_{j-1}T_{\gamma_{j-2}}T_{j-1}(E_j) \\
&= T_{\gamma_1} \cdots T_{\gamma_{j-2}}T_{j-1}T_{\gamma_{j-2}}([E_{j-1}, E_j]_q) \\
&= [T_{\gamma_1} \cdots T_{\gamma_{j-2}}T_{j-1}T_{\gamma_{j-2}}(E_{j-1}), T_{\gamma_1} \cdots T_{\gamma_{j-2}}T_{j-1}(E_j)]_q \\
&= [E_{2-j,j-1}, T_{\gamma_1} \cdots T_{\gamma_{j-2}}([E_{j-1}, E_j]_q)]_q \\
&= [E_{j-1}, [T_{\gamma_1} \cdots T_{\gamma_{j-2}}(E_{j-1}), E_j]_q]_q \\
&= [E_{j-1}, [E_{j-2,j-1}, E_j]_q]_q \\
&= [E_{j-1}, E_{j-2,j}]_q.
\end{aligned}$$

That is, the relation (4.12) holds.

It is easy to check that $T_1T_{\gamma_i} = T_{\gamma_i}T_1$ for any $i \in I^+$, so for $j \geq 2$, using (4.2)–(4.4), (4.5) Proposition 4.2 (III), we obtain

$$\begin{aligned}
E_{j,j} &= T_{\gamma_1} \cdots T_{\gamma_{j-1}}T_jT_{j-1} \cdots T_2(E_1) \\
&= [2]_q^{-1}T_{\gamma_1} \cdots T_{\gamma_{j-1}}T_jT_{j-1} \cdots T_3([E_2, [E_2, E_1]_{q^2}]) \\
&= [2]_q^{-1}[T_{\gamma_1} \cdots T_{\gamma_{j-1}}T_jT_{j-1} \cdots T_3(E_2), T_{\gamma_1} \cdots T_{\gamma_{j-1}}T_jT_{j-1} \cdots T_3([E_2, E_1]_{q^2})] \\
&= [2]_q^{-1}[E_{1,j}, T_{\gamma_1} \cdots T_{\gamma_{j-1}}T_jT_{j-1} \cdots T_3([E_2, E_1]_{q^2})] \\
&= [2]_q^{-1}[E_{1,j}, T_{\gamma_2} \cdots T_{\gamma_{j-1}}T_jT_{j-1} \cdots T_3T_1([E_2, E_1]_{q^2})] \\
&= [2]_q^{-1}[E_{1,j}, T_{\gamma_2} \cdots T_{\gamma_{j-1}}T_jT_{j-1} \cdots T_3(E_2)] \\
&= [2]_q^{-1}[E_{1,j}, T_{\gamma_2} \cdots T_{\gamma_{j-1}}T_jT_{j-1} \cdots T_3T_1T_2T_1(E_2)] \\
&= [2]_q^{-1}[E_{1,j}, T_1T_{\gamma_2} \cdots T_{\gamma_{j-1}}T_jT_{j-1} \cdots T_3T_2T_1(E_2)] \\
&= [2]_q^{-1}[E_{1,j}, E_{-1,j}].
\end{aligned}$$

This proves (4.13). \square

Remark 4.7. By Proposition 4.6, we can perform a double induction first on i then on j with $1 \leq i \leq j \leq n$ to obtain all the positive root vectors $E_{\pm i,j}$ from simple root vectors.

5. Realization of positive root vectors of $U_q(\mathfrak{sp}_{2n})$

5.1. In order to realize all the positive root vectors of $U_q(\mathfrak{sp}_{2n})$ directly and concisely as certain operators in $\text{Diff}(\mathcal{X})$, we introduce some new operators.

Definition 5.1. For $i \in I^+$, set

$$\begin{aligned}
\Lambda_0 &= \tau_{n+1} = \tau_{-n-1} := 1, \\
\Lambda_{-i} &:= \prod_{j=-i}^{-1} \mu_j, \quad \Lambda_i := \prod_{j=1}^i \mu_j, \\
\mathfrak{D}_{-i} &:= \mu_i \tau_{-i-1}^{-1} \partial_{-i}, \quad \mathfrak{D}_i := \tau_1^{-1} \Lambda_{i-1}^{-1} \partial_i,
\end{aligned}$$

$$\mathfrak{X}_{-i_L} := \mu_i^{-1} \mu_{-i} x_{-i_L}, \quad \mathfrak{X}_{i_R} := \Lambda_i^2 x_{i_R},$$

and

$$\begin{aligned} \Phi_0 &:= 0, \quad \Psi_{n+1} := 0, \\ \Phi_i &:= \sum_{j=1}^i q^{j-i} \Lambda_{j-1}^2 \mathfrak{D}_{-j} \mathfrak{D}_j, \quad \Psi_i = \tau_{-i}^2 \sum_{j=i}^n q^{j-i} \tau_{-j}^{-2} \mathfrak{X}_{-j_L} \mathfrak{X}_{j_R}, \\ \mathfrak{X}_{-i_R} &:= q^i \Lambda_{1-i}^2 (\mu_i^2 \mathfrak{X}_{-i_L} + \lambda \mu_{-i}^2 \Psi_{i+1} \mathfrak{D}_i). \end{aligned}$$

Then we get

$$(5.1) \quad \Psi_i = \mathfrak{X}_{-i_L} \mathfrak{X}_{i_R} + q \mu_{-i}^2 \Psi_{i+1},$$

$$(5.2) \quad \Phi_i = \Lambda_{i-1}^2 \mathfrak{D}_{-i} \mathfrak{D}_i + q^{-1} \Phi_{i-1}.$$

The commutation relations in the following three lemmas will be used frequently in this section.

Lemma 5.2. (1) For $k, l \in I$ and $i \in I^+$, we have

$$\begin{aligned} \mathfrak{D}_k \mu_l &= q^{\delta_{kl}} \mu_l \mathfrak{D}_k, \\ \mathfrak{X}_{i_R} \mu_k &= q^{-\delta_{i,k}} \mu_k \mathfrak{X}_{i_R}, \\ \mathfrak{X}_{-i_L} \mu_k &= q^{-\delta_{-i,k}} \mu_k \mathfrak{X}_{-i_L}. \end{aligned}$$

(2) For $i, j \in I^+$ with $i < j$, we have

$$\begin{aligned} [\mathfrak{D}_j, \mathfrak{D}_i]_q &= [\mathfrak{D}_{-i}, \mathfrak{D}_{-j}]_q = 0, \\ [\mathfrak{X}_{j_R}, \mathfrak{X}_{i_R}]_q &= [\mathfrak{X}_{-i_L}, \mathfrak{X}_{-j_L}]_q = 0, \\ [\mathfrak{X}_{i_R}, \mathfrak{D}_j]_q &= [\mathfrak{X}_{-j_L}, \mathfrak{D}_{-i}]_q = 0, \\ [\mathfrak{D}_i, \mathfrak{X}_{j_R}]_q &= [\mathfrak{D}_{-j}, \mathfrak{X}_{-i_L}]_q = 0. \end{aligned}$$

(3) For $i, j \in I^+$ with $i \neq j$, we have

$$\begin{aligned} [\mathfrak{D}_i, \mathfrak{D}_{-j}] &= [\mathfrak{X}_{-i_L}, \mathfrak{X}_{j_R}] = 0, \\ [\mathfrak{D}_i, \mathfrak{X}_{-j_L}] &= [\mathfrak{D}_{-i}, \mathfrak{X}_{j_R}] = 0, \\ [\mathfrak{D}_i, \mathfrak{D}_{-i}]_q &= [\mathfrak{X}_{i_R}, \mathfrak{X}_{-i_L}]_q = 0, \\ [\mathfrak{X}_{-i_L}, \mathfrak{D}_i]_q &= [\mathfrak{D}_{-i}, \mathfrak{X}_{i_R}]_q = 0. \end{aligned}$$

(4) For $i \in I^+$, we have

$$\begin{aligned} \mathfrak{D}_i \mathfrak{X}_{i_R} &= q \lambda^{-1} (q^2 \mu_i^2 - 1), \quad \mathfrak{X}_{i_R} \mathfrak{D}_i = q \lambda^{-1} (\mu_i^2 - 1), \\ \mathfrak{D}_{-i} \mathfrak{X}_{-i_L} &= \lambda^{-1} (q^2 \mu_{-i}^2 - 1), \quad \mathfrak{X}_{-i_L} \mathfrak{D}_{-i} = \lambda^{-1} (\mu_{-i}^2 - 1). \end{aligned}$$

Then

$$\begin{aligned} [\mathfrak{D}_i, \mathfrak{X}_{i_R}] &= q^2 \mu_i^2, \quad [\mathfrak{D}_i, \mathfrak{X}_{i_R}]_{q^2} = q^2, \quad [\mathfrak{X}_{i_R}, \mathfrak{D}_i]_{q^{-2}} = -1, \\ [\mathfrak{D}_{-i}, \mathfrak{X}_{-i_L}] &= q \mu_{-i}^2, \quad [\mathfrak{D}_{-i}, \mathfrak{X}_{-i_L}]_{q^2} = q, \quad [\mathfrak{X}_{-i_L}, \mathfrak{D}_{-i}]_{q^{-2}} = -q^{-1}. \end{aligned}$$

PROOF. Applying both sides of each identity to any normal monomial x^α , using (2.17) and Definitions 3.1 and 5.1, we can obtain these commutation relations. \square

By Definition 5.1, Lemmas 2.1 and 5.2 and (5.2) it is easy to check the following Lemma.

Lemma 5.3. *The operators Φ_i and Ψ_i satisfy the following commutation relations.*

(1) *For $i \in I^+$ and $t, k, l \in I$ with $|t| < i$ and $|k| > i$, we have*

$$\begin{aligned} [\Psi_i, \mu_t] &= [\Phi_i, \mu_k] = 0, \\ [\Psi_i, \mu_k]_{q^{-1}} &= [\Phi_i, \mu_t]_q = 0, \\ [\Psi_i, \mu_{\pm i}]_{q^{-1}} &= [\Phi_i, \mu_{\pm i}]_q = 0. \end{aligned}$$

(2) *For $i, j \in I^+$ with $i < j$, we have*

$$\begin{aligned} [\mathfrak{D}_i, \Psi_j]_q &= [\mathfrak{D}_{-i}, \Psi_j]_{q^{-1}} = 0, \\ [\Psi_j, \mathfrak{X}_{-i_L}]_{q^{-1}} &= [\Psi_j, \mathfrak{X}_{i_R}]_q = 0, \\ [\Phi_i, \mathfrak{X}_{-j_L}]_{q^{-1}} &= [\Phi_i, \mathfrak{X}_{j_R}]_q = 0. \end{aligned}$$

(3) *For $i, j \in I^+$ with $i \leq j$, we have*

$$\begin{aligned} [\Phi_i, \mathfrak{D}_j]_{q^{-1}} &= [\Phi_i, \mathfrak{D}_{-j}]_q = 0, \\ [\Psi_i, \mathfrak{X}_{j_R} \mathfrak{D}_{j+1}] &= -q^{j+2-i} \tau_{-i}^2 \tau_{-j-1}^{-2} \mathfrak{X}_{-j-1_L} \mathfrak{X}_{j_R}, \\ [\Psi_i, \mathfrak{X}_{-j-1_L} \mathfrak{D}_{-j}] &= -q^{j+1-i} \tau_{-i}^2 \tau_{-j-1}^{-2} \mathfrak{X}_{-j-1_L} \mathfrak{X}_{j_R}, \end{aligned}$$

so

$$(5.3) \quad [\Psi_i, \mathfrak{X}_{j_R} \mathfrak{D}_{j+1}] = q[\Psi_i, \mathfrak{X}_{-j-1_L} \mathfrak{D}_{-j}].$$

(4) *For $i \in I^+$ we have*

$$(5.4) \quad \begin{aligned} [\mathfrak{D}_i, \Psi_i]_q &= q \mathfrak{X}_{-i_L}, \\ [\Psi_i, \mathfrak{X}_{-i_L}]_q &= 0, \\ [\Phi_i, \mathfrak{X}_{i_R}]_q &= q^2 \Lambda_i^2 \mathfrak{D}_{-i}, \\ [\Phi_i, \mathfrak{X}_{-i_L}]_{q^{-1}} &= \Lambda_{i-1}^2 \mu_{-i}^2 \mathfrak{D}_i, \\ [\Phi_i, \mathfrak{X}_{-i_L}]_q &= \Lambda_{i-1}^2 \mathfrak{D}_i - \lambda q^{-1} \mathfrak{X}_{-i_L} \Phi_{i-1}. \end{aligned}$$

(5) *For $i, k \in I^+$, we have*

$$(5.5) \quad [\mathfrak{D}_k, [\mathfrak{D}_k, \Psi_i]_q]_{q^{-1}} = 0.$$

We will use Lemma 2.1 freely till the end of this section without mentioned.

Lemma 5.4. *The operators \mathfrak{X}_{-i_R} for $i \in I^+$ satisfy the following commutation relations.*

(1) *For $i \in I^+$, $k, l \in I$ with $|k| < i$ and $|l| \neq i$, we have*

$$(5.6) \quad [\mathfrak{X}_{-i_R}, \mu_k] = [\mathfrak{X}_{-i_R}, \mu_l \mu_{-l}^{-1}] = [\mathfrak{X}_{-i_R}, \mu_i \mu_{-i}^{-1}]_q = 0.$$

(2) *For $i \in I^+$, we have*

$$\begin{aligned} (5.7) \quad [\mathfrak{D}_i, \mathfrak{X}_{-i_R}]_q &= [\mathfrak{X}_{-i_R}, \mathfrak{X}_{-i_L}] = 0, \\ (5.8) \quad [\mathfrak{X}_{-i_R}, \mathfrak{X}_{i_R} \mathfrak{D}_{i+1}]_q &= q^2 \mathfrak{X}_{-i-1_R} - q^{i+3} \Lambda_{-i}^2 \mu_i^2 \mathfrak{X}_{-i-1_L}, \\ (5.9) \quad [\mathfrak{X}_{-i_R}, \mathfrak{X}_{-i-1_L} \mathfrak{D}_{-i}]_q &= -q^{i+2} \Lambda_{-i}^2 \mu_i^2 \mathfrak{X}_{-i-1_L}. \end{aligned}$$

(3) *For $i, j \in I^+$ with $i < j$, we have*

$$(5.10) \quad \begin{aligned} [\mathfrak{X}_{i_R}, \mathfrak{X}_{-j_R}] &= [\mathfrak{X}_{-j_R}, \mathfrak{X}_{-i_L}]_q = 0, \\ [\mathfrak{D}_i, \mathfrak{X}_{-j_R}] &= [\mathfrak{D}_{-i}, \mathfrak{X}_{-j_R}]_q = 0. \end{aligned}$$

PROOF. Using Lemmas 5.2 and 5.3, we can prove this lemma directly. We only show (5.8) for example. For $i \in I^+$ by definition of \mathfrak{X}_{-i_R} , Lemma 5.2 and (5.1) we have

$$\begin{aligned}
[\mathfrak{X}_{-i_R}, \mathfrak{X}_{i_R} \mathfrak{D}_{i+1}]_q &= [q^i \Lambda_{1-i}^2 (\mu_i^2 \mathfrak{X}_{-i_L} + \lambda \mu_{-i}^2 \Psi_{i+1} \mathfrak{D}_i), \mathfrak{X}_{i_R} \mathfrak{D}_{i+1}]_q \\
&= q^i \Lambda_{1-i}^2 [\mu_i^2 \mathfrak{X}_{-i_L}, \mathfrak{X}_{i_R} \mathfrak{D}_{i+1}]_q + q^i \lambda \Lambda_{1-i}^2 [\mu_{-i}^2 \Psi_{i+1} \mathfrak{D}_i, \mathfrak{X}_{i_R} \mathfrak{D}_{i+1}]_q \\
&= q^i \Lambda_{1-i}^2 \mu_i^2 [\mathfrak{X}_{-i_L}, \mathfrak{X}_{i_R} \mathfrak{D}_{i+1}]_{q^{-1}} + q^i \lambda \Lambda_{1-i}^2 \mu_{-i}^2 [\Psi_{i+1} \mathfrak{D}_i, \mathfrak{X}_{i_R} \mathfrak{D}_{i+1}]_q \\
&= 0 + q^i \lambda \Lambda_{-i}^2 [(\mathfrak{X}_{-i-1_L} \mathfrak{X}_{i+1_R} + q \mu_{-i-1}^2 \Psi_{i+2}) \mathfrak{D}_i, \mathfrak{X}_{i_R} \mathfrak{D}_{i+1}]_q \\
&= q^i \lambda \Lambda_{-i}^2 \mathfrak{X}_{-i-1_L} (\mathfrak{X}_{i+1_R} \mathfrak{D}_i \mathfrak{X}_{i_R} \mathfrak{D}_{i+1} - \mathfrak{X}_{i_R} \mathfrak{D}_{i+1} \mathfrak{X}_{i+1_R} \mathfrak{D}_i) \\
&\quad + q^{i+1} \lambda \Lambda_{-1-i}^2 \Psi_{i+2} [\mathfrak{D}_i, \mathfrak{X}_{i_R}]_{q^2} \mathfrak{D}_{i+1} \\
&= q^i \lambda \Lambda_{-i}^2 \mathfrak{X}_{-i-1_L} (\mathfrak{D}_i \mathfrak{X}_{i_R} \mathfrak{X}_{i+1_R} \mathfrak{D}_{i+1} - \mathfrak{X}_{i_R} \mathfrak{D}_i \mathfrak{D}_{i+1} \mathfrak{X}_{i+1_R}) \\
&\quad + q^{i+3} \lambda \Lambda_{-1-i}^2 \Psi_{i+2} \mathfrak{D}_{i+1} \\
&= q^i \lambda \Lambda_{-i}^2 \mathfrak{X}_{-i-1_L} ([\mathfrak{D}_i, \mathfrak{X}_{i_R}] \mathfrak{X}_{i+1_R} \mathfrak{D}_{i+1} - \mathfrak{X}_{i_R} \mathfrak{D}_i [\mathfrak{D}_{i+1}, \mathfrak{X}_{i+1_R}]) \\
&\quad + q^{i+3} \lambda \Lambda_{-1-i}^2 \Psi_{i+2} \mathfrak{D}_{i+1} \\
&= q^{i+3} \Lambda_{-i}^2 \mathfrak{X}_{-i-1_L} (\mu_i^2 (\mu_{i+1}^2 - 1) - (\mu_i^2 - 1) \mu_{i+1}^2) \\
&\quad + q^{i+3} \lambda \Lambda_{-1-i}^2 \Psi_{i+2} \mathfrak{D}_{i+1} \\
&= q^{i+3} \Lambda_{-i}^2 (\mu_{i+1}^2 - \mu_i^2) \mathfrak{X}_{-i-1_L} + q^{i+3} \lambda \Lambda_{-1-i}^2 \Psi_{i+2} \mathfrak{D}_{i+1} \\
&= q^2 \mathfrak{X}_{-i-1_R} - q^{i+3} \Lambda_{-i}^2 \mu_i^2 \mathfrak{X}_{-i-1_L},
\end{aligned}$$

that is, (5.8) holds. \square

Corollary 5.5. For $i, j \in I^+$ with $i \leq j$, we have

$$(5.11) \quad [\mathfrak{X}_{-j_R}, [\mathfrak{D}_j, \Psi_i]_q] = 0.$$

PROOF. For $1 \leq i \leq j \leq n$, Lemma 5.2 yields

$$\begin{aligned}
[\mathfrak{D}_j, \Psi_i]_q &= [\mathfrak{D}_j, \tau_{-i}^2 \sum_{k=i}^n q^{k-i} \tau_{-k}^{-2} \mathfrak{X}_{-k_L} \mathfrak{X}_{k_R}]_q = \tau_{-i}^2 \sum_{k=i}^n q^{k-i} \tau_{-k}^{-2} [\mathfrak{D}_j, \mathfrak{X}_{-k_L} \mathfrak{X}_{k_R}]_q \\
&= -\lambda \tau_{-i}^2 \sum_{k=i}^{j-1} q^{k-i} \tau_{-k}^{-2} \mathfrak{X}_{-k_L} \mathfrak{X}_{k_R} \mathfrak{D}_j + q^{j-i-1} \tau_{-i}^2 \tau_{-j}^{-2} \mathfrak{X}_{-j_L} [\mathfrak{D}_j, \mathfrak{X}_{j_R}]_{q^2} \\
&= -\lambda \sum_{k=i}^{j-1} q^{k-i} \tau_{-i}^2 \tau_{-k}^{-2} \mathfrak{X}_{-k_L} \mathfrak{X}_{k_R} \mathfrak{D}_j + q^{j-i+1} \tau_{-i}^2 \tau_{-j}^{-2} \mathfrak{X}_{-j_L},
\end{aligned}$$

then by (5.6), (5.7) and (5.10)

$$\begin{aligned}
&[\mathfrak{X}_{-j_R}, [\mathfrak{D}_j, \Psi_i]_q] \\
&= -\lambda \sum_{k=i}^{j-1} q^{k-i} \tau_{-i}^2 \tau_{-k}^{-2} [\mathfrak{X}_{-j_R}, \mathfrak{X}_{-k_L} \mathfrak{X}_{k_R} \mathfrak{D}_j] + q^{j-i+1} \tau_{-i}^2 \tau_{-j}^{-2} [\mathfrak{X}_{-j_R}, \mathfrak{X}_{-j_L}] \\
&= -\lambda \sum_{k=i}^{j-1} q^{k-i} \tau_{-i}^2 \tau_{-k}^{-2} [\mathfrak{X}_{-j_R}, \mathfrak{X}_{-k_L}]_q \mathfrak{X}_{k_R} \mathfrak{D}_j = 0,
\end{aligned}$$

that is, (5.11) holds. \square

We are now in the position to realize all the positive root vectors $E_{\pm i, j}$ of $U_q(\mathfrak{sp}_{2n})$ as $e_{\pm i, j}$ in $\text{Diff}(\mathcal{X})$.

Definition 5.6. For $i, j \in I^+$ with $i < j$, set

$$\begin{aligned} e_{-i, j} &:= (-1)^{i+j} q^{-2} (\mathfrak{X}_{i_R} \mathfrak{D}_j - [\mathfrak{D}_j, \Psi_{i+1}]_q \mathfrak{D}_{-i}), \\ e_{i, i} &:= [2]_q^{-1} \tau_1 \tau_{-1}^{-1} (\mathfrak{X}_{-i_R} \mathfrak{D}_i + q^{-2} [\mathfrak{D}_i, \Psi_1]_q \mathfrak{D}_i), \\ e_{i, j} &:= (-1)^{j+1} \tau_1 \tau_{-1}^{-1} (\mathfrak{X}_{-i_L} \mathfrak{D}_j + q^{i-1} \mathfrak{X}_{-j_R} [\Phi_i, \mathfrak{X}_{-i_L}]_q). \end{aligned}$$

The operators defined in Definition 5.6 coincide with those defined in Definition 3.2.

Lemma 5.7. $e_{1,1} = e_1$ and $e_{1-i, i} = e_i$ for $1 < i \leq n$.

PROOF. From (2.20) it is easy to show that

$$\Omega_i x^\alpha = \tau_1^{-1} \tau_{-1} \tau_{-i}^2 \sum_{j=i}^n q^{j-i-2} \tau_{-j}^{-2} \mathfrak{X}_{-j_L} \mathfrak{X}_{j_R} x^\alpha$$

for any normal monomial x^α . Then it yields from (2.19) that

$$\begin{aligned} x_{-i_R} &= \tau_1 \tau_{-i}^{-1} \mu_i^2 \Lambda_{1-i} \mathfrak{X}_{-i_L} + \lambda \tau_1 \tau_{-1} \sum_{j=i+1}^n q^{j-i-1} \tau_{-j}^{-2} \mathfrak{X}_{-j_L} \mathfrak{X}_{j_R} \mathfrak{D}_i \\ &= \tau_1 \tau_{-i}^{-1} \mu_i^2 \Lambda_{1-i} \mathfrak{X}_{-i_L} + \lambda \tau_1 \Lambda_{-i} \tau_{-i-1}^{-1} \Psi_{i+1} \mathfrak{D}_i. \end{aligned}$$

Write e_i in terms of the new operators defined in Definition 5.6. By (5.4) we get

$$\begin{aligned} e_1 &= [2]_q^{-1} q^{-1} \mu_1^{-1} (\tau_{-2}^{-1} x_{-1_L} + q^2 \tau_2^{-1} x_{-1_R}) \partial_1 \\ &= [2]_q^{-1} \tau_{-1}^{-1} (q^{-1} \mathfrak{X}_{-1_L} + q \mu_1^2 \mathfrak{X}_{-1_L} + q \lambda \mu_{-1}^2 \Psi_2 \mathfrak{D}_1) \tau_1 \mathfrak{D}_1 \\ &= [2]_q^{-1} \tau_1 \tau_{-1}^{-1} (q^{-1} \mathfrak{X}_{-1_L} + \mathfrak{X}_{-1_R}) \mathfrak{D}_1 \\ &= [2]_q^{-1} \tau_1 \tau_{-1}^{-1} (q^{-2} [\mathfrak{D}_1, \Psi_1]_q + \mathfrak{X}_{-1_R}) \mathfrak{D}_1 \\ &= e_{1,1}, \end{aligned}$$

and for $i > 1$

$$\begin{aligned} e_i &= \mu_{i-1} \mu_i^{-1} \tau_{-i-1}^{-1} x_{-i_L} \partial_{1-i} - \tau_i^{-1} x_{i-1_R} \partial_i \\ &= q^{-1} \mathfrak{X}_{-i_L} \mathfrak{D}_{1-i} - q^{-2} \mathfrak{X}_{i-1_R} \mathfrak{D}_i \\ &= q^{-2} ([\mathfrak{D}_i, \Psi_i]_q \mathfrak{D}_{1-i} - \mathfrak{X}_{i-1_R} \mathfrak{D}_i) \\ &= e_{1-i, i}. \end{aligned}$$

□

Proposition 5.8. The commutation relations (4.9)–(4.13) remain valid if E is replaced by e .

PROOF. (1) To prove $e_{1,2} = [e_1, e_2]_{q^2}$, we compute the following four brackets first. By (2.3), (5.1), (5.8), (5.9) and Lemma 5.2 we get

$$\begin{aligned} (4.9'a) \quad & [\mathfrak{X}_{-1_R} \mathfrak{D}_1, \mathfrak{X}_{1_R} \mathfrak{D}_2]_{q^2} = \mathfrak{X}_{-1_R} [\mathfrak{D}_1, \mathfrak{X}_{1_R} \mathfrak{D}_2]_q + q [\mathfrak{X}_{-1_R}, \mathfrak{X}_{1_R} \mathfrak{D}_2]_q \mathfrak{D}_1 \\ &= \mathfrak{X}_{-1_R} [\mathfrak{D}_1, \mathfrak{X}_{1_R}]_{q^2} \mathfrak{D}_2 + q (q^2 \mathfrak{X}_{-2_R} - q^4 \Lambda_{-1}^2 \mu_1^2 \mathfrak{X}_{-2_L}) \mathfrak{D}_1 \\ &= q^2 \mathfrak{X}_{-1_R} \mathfrak{D}_2 + q^3 \mathfrak{X}_{-2_R} \mathfrak{D}_1 - q^5 \mu_{-1}^2 \mu_1^2 \mathfrak{X}_{-2_L} \mathfrak{D}_1 \\ &= q^3 (\mu_1^2 \mathfrak{X}_{-1_L} + \lambda \mu_{-1}^2 \Psi_2 \mathfrak{D}_1) \mathfrak{D}_2 + q^3 \mathfrak{X}_{-2_R} \mathfrak{D}_1 - q^5 \mu_{-1}^2 \mu_1^2 \mathfrak{X}_{-2_L} \mathfrak{D}_1 \end{aligned}$$

$$\begin{aligned}
&= q^3 \mu_1^2 \mathfrak{X}_{-1_L} \mathfrak{D}_2 + q^2 \lambda \mu_{-1}^2 (\mathfrak{X}_{-2_L} \mathfrak{X}_{2_R} + q \mu_{-2}^2 \Psi_3) \mathfrak{D}_2 \mathfrak{D}_1 \\
&\quad + q^3 \mathfrak{X}_{-2_R} \mathfrak{D}_1 - q^5 \mu_{-1}^2 \mu_1^2 \mathfrak{X}_{-2_L} \mathfrak{D}_1 \\
&= q^3 \mu_1^2 \mathfrak{X}_{-1_L} \mathfrak{D}_2 + q^3 \mu_{-1}^2 (\mu_2^2 - 1) \mathfrak{X}_{-2_L} \mathfrak{D}_1 + q^3 \lambda \mu_{-1}^2 \mu_{-2}^2 \Psi_3 \mathfrak{D}_2 \mathfrak{D}_1 \\
&\quad + q^3 \mathfrak{X}_{-2_R} \mathfrak{D}_1 - q^5 \mu_{-1}^2 \mu_1^2 \mathfrak{X}_{-2_L} \mathfrak{D}_1 \\
&= q^3 \mu_1^2 \mathfrak{X}_{-1_L} \mathfrak{D}_2 - q^3 \mu_{-1}^2 \mathfrak{X}_{-2_L} \mathfrak{D}_1 - q^5 \mu_{-1}^2 \mu_1^2 \mathfrak{X}_{-2_L} \mathfrak{D}_1 + q^2 [2]_q \mathfrak{X}_{-2_R} \mathfrak{D}_1, \\
(4.9'b) \quad &[q^{-1} \mathfrak{X}_{-1_L} \mathfrak{D}_1, \mathfrak{X}_{1_R} \mathfrak{D}_2]_{q^2} = q^{-1} \mathfrak{X}_{-1_L} [\mathfrak{D}_1, \mathfrak{X}_{1_R}]_{q^4} \mathfrak{D}_2 \\
&= \lambda^{-1} \mathfrak{X}_{-1_L} ((q^2 \mu_1^2 - 1) - q^4 (\mu_1^2 - 1)) \mathfrak{D}_2 \\
&= q \mathfrak{X}_{-1_L} (q^2 + 1 - q^2 \mu_1^2) \mathfrak{D}_2 \\
&= q(q[2]_q - q^2 \mu_1^2) \mathfrak{X}_{-1_L} \mathfrak{D}_2 \\
&= q^2 [2]_q \mathfrak{X}_{-1_L} \mathfrak{D}_2 - q^3 \mu_1^2 \mathfrak{X}_{-1_L} \mathfrak{D}_2, \\
(4.9'c) \quad &[\mathfrak{X}_{-1_R} \mathfrak{D}_1, -q \mathfrak{X}_{-2_L} \mathfrak{D}_{-1}]_{q^2} \\
&= -q \mathfrak{X}_{-1_R} [\mathfrak{D}_1, \mathfrak{X}_{-2_L} \mathfrak{D}_{-1}]_q - q^2 [\mathfrak{X}_{-1_R}, \mathfrak{X}_{-2_L} \mathfrak{D}_{-1}]_q \mathfrak{D}_1 \\
&= -q \mathfrak{X}_{-1_R} \mathfrak{X}_{-2_L} [\mathfrak{D}_1, \mathfrak{D}_{-1}]_q + q^5 \mu_{-1}^2 \mu_1^2 \mathfrak{X}_{-2_L} \mathfrak{D}_1 \\
&= q^5 \mu_{-1}^2 \mu_1^2 \mathfrak{X}_{-2_L} \mathfrak{D}_1, \\
(4.9'd) \quad &[q^{-1} \mathfrak{X}_{-1_L} \mathfrak{D}_1, -q \mathfrak{X}_{-2_L} \mathfrak{D}_{-1}]_{q^2} = -q [\mathfrak{X}_{-1_L}, \mathfrak{X}_{-2_L} \mathfrak{D}_{-1}]_q \mathfrak{D}_1 \\
&= -q^2 \mathfrak{X}_{-2_L} [\mathfrak{X}_{-1_L}, \mathfrak{D}_{-1}] \mathfrak{D}_1 = q^3 \mathfrak{X}_{-2_L} \mu_{-1}^2 \mathfrak{D}_1 \\
&= q^3 \mu_{-1}^2 \mathfrak{X}_{-2_L} \mathfrak{D}_1.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
[e_1, e_2]_{q^2} &= [e_{1,1}, e_{-1,2}]_{q^2} \\
&= -q^{-2} [2]_q^{-1} \tau_1 \tau_{-1}^{-1} [\mathfrak{X}_{-1_R} \mathfrak{D}_1 + q^{-1} \mathfrak{X}_{-1_L} \mathfrak{D}_1, \mathfrak{X}_{1_R} \mathfrak{D}_2 - q \mathfrak{X}_{-2_L} \mathfrak{D}_{-1}]_{q^2}.
\end{aligned}$$

So from (4.9'a)–(4.9'd) we obtain

$$[e_1, e_2]_{q^2} = -\tau_1 \tau_{-1}^{-1} (\mathfrak{X}_{-1_L} \mathfrak{D}_2 + \mathfrak{X}_{-2_R} \mathfrak{D}_1) = e_{1,2}.$$

(2) To prove $e_{-i,j} = [e_{-i,j-1}, e_j]_q$ for $3 \leq i+2 \leq j \leq n$, we compute the following four brackets first. By (2.4), (5.3) and Lemma 5.2, for $3 \leq i+2 \leq j \leq n$, we get

$$\begin{aligned}
(4.10'a) \quad &[\mathfrak{X}_{i_R} \mathfrak{D}_{j-1}, \mathfrak{X}_{j-1_R} \mathfrak{D}_j]_q = \mathfrak{X}_{i_R} [\mathfrak{D}_{j-1}, \mathfrak{X}_{j-1_R} \mathfrak{D}_j]_q \\
&= \mathfrak{X}_{i_R} [\mathfrak{D}_{j-1}, \mathfrak{X}_{j-1_R}]_{q^2} \mathfrak{D}_j = q^2 \mathfrak{X}_{i_R} \mathfrak{D}_j,
\end{aligned}$$

$$\begin{aligned}
(4.10'b) \quad &[[\mathfrak{D}_{j-1}, \Psi_{i+1}]_q \mathfrak{D}_{-i}, \mathfrak{X}_{j-1_R} \mathfrak{D}_j]_q = [[\mathfrak{D}_{j-1}, \Psi_{i+1}]_q, \mathfrak{X}_{j-1_R} \mathfrak{D}_j]_q \mathfrak{D}_{-i} \\
&= [\mathfrak{D}_{j-1}, [\Psi_{i+1}, \mathfrak{X}_{j-1_R} \mathfrak{D}_j]]_{q^2} \mathfrak{D}_{-i} + [[\mathfrak{D}_{j-1}, \mathfrak{X}_{j-1_R} \mathfrak{D}_j]_q, \Psi_{i+1}]_q \mathfrak{D}_{-i} \\
&= [\mathfrak{D}_{j-1}, [\Psi_{i+1}, \mathfrak{X}_{j-1_R} \mathfrak{D}_j]]_{q^2} \mathfrak{D}_{-i} + [[\mathfrak{D}_{j-1}, \mathfrak{X}_{j-1_R}]_{q^2} \mathfrak{D}_j, \Psi_{i+1}]_q \mathfrak{D}_{-i} \\
&= [\mathfrak{D}_{j-1}, [\Psi_{i+1}, \mathfrak{X}_{j-1_R} \mathfrak{D}_j]]_{q^2} \mathfrak{D}_{-i} + q^2 [\mathfrak{D}_j, \Psi_{i+1}]_q \mathfrak{D}_{-i},
\end{aligned}$$

$$(4.10'c) \quad [\mathfrak{X}_{i_R} \mathfrak{D}_{j-1}, q \mathfrak{X}_{-j_L} \mathfrak{D}_{1-j}]_q = q [\mathfrak{X}_{i_R} \mathfrak{D}_{j-1}, \mathfrak{X}_{-j_L}] \mathfrak{D}_{1-j} = 0,$$

$$\begin{aligned}
(4.10'd) \quad &[[\mathfrak{D}_{j-1}, \Psi_{i+1}]_q \mathfrak{D}_{-i}, q \mathfrak{X}_{-j_L} \mathfrak{D}_{1-j}]_q = [[\mathfrak{D}_{j-1}, \Psi_{i+1}]_q, q \mathfrak{X}_{-j_L} \mathfrak{D}_{1-j}]_q \mathfrak{D}_{-i} \\
&= q [\mathfrak{D}_{j-1}, [\Psi_{i+1}, \mathfrak{X}_{-j_L} \mathfrak{D}_{1-j}]]_{q^2} \mathfrak{D}_{-i} + q [[\mathfrak{D}_{j-1}, \mathfrak{X}_{-j_L} \mathfrak{D}_{1-j}]_q, \Psi_{i+1}]_q \mathfrak{D}_{-i}
\end{aligned}$$

$$\begin{aligned}
&= [\mathfrak{D}_{j-1}, [\Psi_{i+1}, \mathfrak{X}_{j-1R} \mathfrak{D}_j]]_{q^2} \mathfrak{D}_{-i} + q[[\mathfrak{D}_{j-1}, \mathfrak{X}_{-jL}] \mathfrak{D}_{1-j}, \Psi_{i+1}]_q \mathfrak{D}_{-i} \\
&= [\mathfrak{D}_{j-1}, [\Psi_{i+1}, \mathfrak{X}_{j-1R} \mathfrak{D}_j]]_{q^2} \mathfrak{D}_{-i}.
\end{aligned}$$

From (4.10'a)–(4.10'd) it is easy to see that

$$\begin{aligned}
[e_{-i,j-1}, e_j]_q &= [e_{-i,j-1}, e_{1-j,j}]_q \\
&= (-1)^{i+j} q^{-4} [\mathfrak{X}_{iR} \mathfrak{D}_{j-1} - [\mathfrak{D}_{j-1}, \Psi_{i+1}]_q \mathfrak{D}_{-i}, \mathfrak{X}_{j-1R} \mathfrak{D}_j - q \mathfrak{X}_{-jL} \mathfrak{D}_{1-j}]_q \\
&= (-1)^{i+j} q^{-2} (\mathfrak{X}_{iR} \mathfrak{D}_j - [\mathfrak{D}_j, \Psi_{i+1}]_q \mathfrak{D}_{-i}) \\
&= e_{-i,j}.
\end{aligned}$$

(3) To prove $e_{i,j} = [e_{i,j-1}, e_j]_q$ for $3 \leq i+2 \leq j \leq n$, we compute the following four brackets first. By (2.2), (5.8), (5.9), Lemmas 5.2 and 5.3, for $3 \leq i+2 \leq j \leq n$, we get

$$(4.11'a) \quad [\mathfrak{X}_{-iL} \mathfrak{D}_{j-1}, \mathfrak{X}_{j-1R} \mathfrak{D}_j]_q = \mathfrak{X}_{-iL} [\mathfrak{D}_{j-1}, \mathfrak{X}_{j-1R}]_{q^2} \mathfrak{D}_j = q^2 \mathfrak{X}_{-iL} \mathfrak{D}_j,$$

$$\begin{aligned}
(4.11'b) \quad & [q^{i-1} \mathfrak{X}_{1-jR} [\Phi_i, \mathfrak{X}_{-iL}]_q, \mathfrak{X}_{j-1R} \mathfrak{D}_j]_q \\
&= q^{i-1} [\mathfrak{X}_{1-jR}, \mathfrak{X}_{j-1R} \mathfrak{D}_j]_q [\Phi_i, \mathfrak{X}_{-iL}]_q \\
&= q^{i-1} (q^2 \mathfrak{X}_{-jR} - q^{j+2} \Lambda_{1-j}^2 \mu_{j-1}^2 \mathfrak{X}_{-jL}) [\Phi_i, \mathfrak{X}_{-iL}]_q \\
&= q^{i+1} \mathfrak{X}_{-jR} [\Phi_i, \mathfrak{X}_{-iL}]_q - q^{i+j+1} \Lambda_{1-j}^2 \mu_{j-1}^2 \mathfrak{X}_{-jL} [\Phi_i, \mathfrak{X}_{-iL}]_q,
\end{aligned}$$

$$(4.11'c) \quad [\mathfrak{X}_{-iL} \mathfrak{D}_{j-1}, -q \mathfrak{X}_{-jL} \mathfrak{D}_{1-j}]_q = -q \mathfrak{X}_{-iL} \mathfrak{X}_{-jL} [\mathfrak{D}_{j-1}, \mathfrak{D}_{1-j}]_q = 0,$$

$$\begin{aligned}
(4.11'd) \quad & [q^{i-1} \mathfrak{X}_{1-jR} [\Phi_i, \mathfrak{X}_{-iL}]_q, -q \mathfrak{X}_{-jL} \mathfrak{D}_{1-j}]_q \\
&= -q^i [\mathfrak{X}_{1-jR}, \mathfrak{X}_{-jL} \mathfrak{D}_{1-j}]_q [\Phi_i, \mathfrak{X}_{-iL}]_q \\
&= q^{i+j+1} \Lambda_{1-j}^2 \mu_{j-1}^2 \mathfrak{X}_{-jL} [\Phi_i, \mathfrak{X}_{-iL}]_q.
\end{aligned}$$

From (4.11'a)–(4.11'd) it is easy to see that

$$\begin{aligned}
& [e_{i,j-1}, e_j]_q \\
&= (-1)^{j+1} q^{-2} \tau_1 \tau_{-1}^{-1} [\mathfrak{X}_{-iL} \mathfrak{D}_{j-1} + q^{i-1} \mathfrak{X}_{1-jR} [\Phi_i, \mathfrak{X}_{-iL}]_q, \mathfrak{X}_{j-1R} \mathfrak{D}_j - q \mathfrak{X}_{-jL} \mathfrak{D}_{1-j}]_q \\
&= (-1)^{j+1} \tau_1 \tau_{-1}^{-1} (\mathfrak{X}_{-iL} \mathfrak{D}_j + q^{i-1} \mathfrak{X}_{-jR} [\Phi_i, \mathfrak{X}_{-iL}]_q) \\
&= e_{i,j}.
\end{aligned}$$

(4) To prove $e_{j-1,j} = [e_{j-1}, e_{j-2,j}]_q$ for $3 \leq j \leq n$, we compute the following four brackets first. By (2.2) and Lemmas 5.2–5.4, for $3 \leq j \leq n$, we get

$$\begin{aligned}
(4.12'a) \quad & [\mathfrak{X}_{j-2R} \mathfrak{D}_{j-1}, \mathfrak{X}_{2-jL} \mathfrak{D}_j]_q = q^{-1} [\mathfrak{X}_{j-2R}, \mathfrak{X}_{2-jL} \mathfrak{D}_j]_{q^2} \mathfrak{D}_{j-1} \\
&= q^{-1} [\mathfrak{X}_{j-2R}, \mathfrak{X}_{2-jL}]_q \mathfrak{D}_j \mathfrak{D}_{j-1} = 0,
\end{aligned}$$

$$\begin{aligned}
(4.12'b) \quad & q^{j-3} [\mathfrak{X}_{j-2R} \mathfrak{D}_{j-1}, \mathfrak{X}_{-jR} [\Phi_{j-2}, \mathfrak{X}_{2-jL}]_q]_q \\
&= q^{j-2} \mathfrak{X}_{-jR} [\mathfrak{X}_{j-2R}, [\Phi_{j-2}, \mathfrak{X}_{2-jL}]_q] \mathfrak{D}_{j-1} \\
&= -q^{j-3} \mathfrak{X}_{-jR} [[\Phi_{j-2}, \mathfrak{X}_{j-2R}]_q, \mathfrak{X}_{2-jL}]_{q^2} \mathfrak{D}_{j-1} \\
&= -q^{j-1} \mathfrak{X}_{-jR} [\Lambda_{j-2}^2 \mathfrak{D}_{2-j}, \mathfrak{X}_{2-jL}]_{q^2} \mathfrak{D}_{j-1} \\
&= -q^{j-1} \Lambda_{j-2}^2 \mathfrak{X}_{-jR} [\mathfrak{D}_{2-j}, \mathfrak{X}_{2-jL}]_{q^2} \mathfrak{D}_{j-1} \\
&= -q^j \Lambda_{j-2}^2 \mathfrak{X}_{-jR} \mathfrak{D}_{j-1},
\end{aligned}$$

$$\begin{aligned}
 (4.12'c) \quad & -q[\mathfrak{X}_{1-j_L}\mathfrak{D}_{2-j}, \mathfrak{X}_{2-j_L}\mathfrak{D}_j]_q = -q[\mathfrak{X}_{1-j_L}\mathfrak{D}_{2-j}, \mathfrak{X}_{2-j_L}]_q\mathfrak{D}_j \\
 & = -q\mathfrak{X}_{1-j_L}[\mathfrak{D}_{2-j}, \mathfrak{X}_{2-j_L}]_{q^2}\mathfrak{D}_j = -q^2\mathfrak{X}_{1-j_L}\mathfrak{D}_j,
 \end{aligned}$$

$$\begin{aligned}
 (4.12'd) \quad & -q^{j-2}[\mathfrak{X}_{1-j_L}\mathfrak{D}_{2-j}, \mathfrak{X}_{-j_R}[\Phi_{j-2}, \mathfrak{X}_{2-j_L}]_q]_q \\
 & = -q^{j-2}\mathfrak{X}_{-j_R}[\mathfrak{X}_{1-j_L}\mathfrak{D}_{2-j}, [\Phi_{j-2}, \mathfrak{X}_{2-j_L}]_q]_q \\
 & = -q^{j-2}\mathfrak{X}_{-j_R}\mathfrak{X}_{1-j_L}[\mathfrak{D}_{2-j}, [\Phi_{j-2}, \mathfrak{X}_{2-j_L}]_q]_q \\
 & = q^{j-1}\mathfrak{X}_{-j_R}\mathfrak{X}_{1-j_L}[\Phi_{j-2}, [\mathfrak{X}_{2-j_L}, \mathfrak{D}_{2-j}]_{q^{-2}}]_{q^2} \\
 & = q^{j-1}\mathfrak{X}_{-j_R}\mathfrak{X}_{1-j_L}[\Phi_{j-2}, -q^{-1}]_{q^2} \\
 & = q^{j-1}\lambda\mathfrak{X}_{-j_R}\mathfrak{X}_{1-j_L}\Phi_{j-2}.
 \end{aligned}$$

From (4.12'a)–(4.12'd) it is easy to see that

$$\begin{aligned}
 & [e_{j-1}, e_{j-2,j}]_q \\
 & = (-1)^j q^{-2}\tau_1\tau_{-1}^{-1}[\mathfrak{X}_{j-2_R}\mathfrak{D}_{j-1} - q\mathfrak{X}_{1-j_L}\mathfrak{D}_{2-j}, \mathfrak{X}_{2-j_L}\mathfrak{D}_j + q^{j-3}\mathfrak{X}_{-j_R}[\Phi_{j-2}, \mathfrak{X}_{2-j_L}]_q]_q \\
 & = (-1)^{j+1}\tau_1\tau_{-1}^{-1}(\mathfrak{X}_{1-j_L}\mathfrak{D}_j + q^{j-2}\mathfrak{X}_{-j_R}(\Lambda_{j-2}^2\mathfrak{D}_{j-1} - \lambda q^{-1}\mathfrak{X}_{1-j_L}\Phi_{j-2})) \\
 & = (-1)^{j+1}\tau_1\tau_{-1}^{-1}(\mathfrak{X}_{1-j_L}\mathfrak{D}_j + q^{j-2}\mathfrak{X}_{-j_R}[\Phi_{j-1}, \mathfrak{X}_{1-j_L}]_q) \\
 & = e_{j-1,j}.
 \end{aligned}$$

(5) To prove $e_{j,j} = [2]_q^{-1}[e_{1,j}, e_{-1,j}]$ for $2 \leq j \leq n$, we compute the following four brackets first. By Lemma 5.2, (5.1), (5.5) and (5.11), for $2 \leq j \leq n$, we get

$$(4.13'a) \quad [\mathfrak{X}_{-1_L}\mathfrak{D}_j, \mathfrak{X}_{1_R}\mathfrak{D}_j] = q^{-1}[\mathfrak{X}_{-1_L}, \mathfrak{X}_{1_R}]_q\mathfrak{D}_j^2 = -\lambda\mathfrak{X}_{-1_L}\mathfrak{X}_{1_R}\mathfrak{D}_j^2,$$

$$\begin{aligned}
 (4.13'b) \quad & -[\mathfrak{X}_{-1_L}\mathfrak{D}_j, [\mathfrak{D}_j, \Psi_2]_q\mathfrak{D}_{-1}] = -q^{-1}[\mathfrak{X}_{-1_L}, [\mathfrak{D}_j, \Psi_2]_q\mathfrak{D}_{-1}]_q\mathfrak{D}_j \\
 & = -[\mathfrak{D}_j, \Psi_2]_q[\mathfrak{X}_{-1_L}, \mathfrak{D}_{-1}]\mathfrak{D}_j = q\mu_{-1}^2[\mathfrak{D}_j, \Psi_2]_q\mathfrak{D}_j \\
 & = q\mu_{-1}^2[\mathfrak{D}_j, q^{-1}\mu_{-1}^{-2}(\Psi_1 - \mathfrak{X}_{-1_L}\mathfrak{X}_{1_R})]_q\mathfrak{D}_j \\
 & = [\mathfrak{D}_j, \Psi_1]_q\mathfrak{D}_j + \lambda\mathfrak{X}_{-1_L}\mathfrak{X}_{1_R}\mathfrak{D}_j^2,
 \end{aligned}$$

$$\begin{aligned}
 (4.13'c) \quad & [\mathfrak{X}_{-j_R}\mathfrak{D}_1, \mathfrak{X}_{1_R}\mathfrak{D}_j] = \mathfrak{X}_{-j_R}[\mathfrak{D}_1, \mathfrak{X}_{1_R}\mathfrak{D}_j]_q \\
 & = \mathfrak{X}_{-j_R}[\mathfrak{D}_1, \mathfrak{X}_{1_R}]_{q^2}\mathfrak{D}_j = q^2\mathfrak{X}_{-j_R}\mathfrak{D}_j,
 \end{aligned}$$

$$\begin{aligned}
 (4.13'd) \quad & -[\mathfrak{X}_{-j_R}\mathfrak{D}_1, [\mathfrak{D}_j, \Psi_2]_q\mathfrak{D}_{-1}] = -q[\mathfrak{X}_{-j_R}, [\mathfrak{D}_j, \Psi_2]_q\mathfrak{D}_{-1}]_{q^{-1}}\mathfrak{D}_1 \\
 & = -q[\mathfrak{X}_{-j_R}, [\mathfrak{D}_j, \Psi_2]_q]\mathfrak{D}_{-1}\mathfrak{D}_1 = 0.
 \end{aligned}$$

It is easy to see that

$$\begin{aligned}
 [e_{1,j}, e_{-1,j}] & = (-1)^{i+j}q^{-2}[\tau_1\tau_{-1}^{-1}(\mathfrak{X}_{-1_L}\mathfrak{D}_j + \mathfrak{X}_{-j_R}\mathfrak{D}_1), \mathfrak{X}_{1_R}\mathfrak{D}_j - [\mathfrak{D}_j, \Psi_2]_q\mathfrak{D}_{-1}] \\
 & = (-1)^{i+j}q^{-2}\tau_1\tau_{-1}^{-1}[\mathfrak{X}_{-1_L}\mathfrak{D}_j + \mathfrak{X}_{-j_R}\mathfrak{D}_1, \mathfrak{X}_{1_R}\mathfrak{D}_j - [\mathfrak{D}_j, \Psi_2]_q\mathfrak{D}_{-1}].
 \end{aligned}$$

So from (4.13'a)–(4.13'd) we get

$$[e_{1,j}, e_{-1,j}] = (-1)^{i+j}\tau_1\tau_{-1}^{-1}(\mathfrak{X}_{-j_R}\mathfrak{D}_j + q^{-2}[\mathfrak{D}_j, \Psi_1]_q\mathfrak{D}_j) = [2]_qe_{j,j}.$$

□

Hence the operators $e_{\pm i,j}$ defined in Definition 5.6 can be obtained inductively by e_i defined in Definition 3.2. Therefore, by Theorem 3.6 and Propositions 4.6 and 5.8, all the positive root vectors $E_{\pm i,j}$ of $U_q(\mathfrak{sp}_{2n})$ can be realized by the operators $e_{\pm i,j}$ in the subalgebra U_q^{2n} of $\text{Diff}(\mathcal{X})$.

References

- [1] V. Chari and Nanhua Xi, *Monomial bases of quantized enveloping algebras*, Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), 69–81, Contemp. Math., **248**, Amer. Math. Soc., Providence, RI, 1999.
- [2] G. Fiore, *Realization of $U_q(\mathfrak{so}(N))$ within the differential algebra on R_q^N* , Comm. Math. Phys. **169** (1995), 475–500.
- [3] W. Fulton and J. Harris, “*Representation Theory, A First Course*”, Graduate Texts in Math., vol. **129**, Springer-Verlag, New York/Heidelberg/Berlin, 1991.
- [4] Haixia Gu and Naihong Hu, *Loewy filtration and quantum de Rham cohomology over quantum divided power algebra*, J. Algebra **435** (2015), 1–32.
- [5] I. Heckenberger, *Spin geometry on quantum groups via covariant differential calculi*, Adv. Math. **175** (2) (2003), 197–242.
- [6] Hongmei Hu and Naihong Hu, *Double-bosonization and Majid’s conjecture, (I): Rank-inductions of ABCD*, J. Math. Phys. **56** (11) (2015), 111702-1, 16 pp.
- [7] Naihong Hu, *Quantum divided power algebra, q -derivatives, and some new quantum groups*, J. Algebra **232** (2000), 507–540.
- [8] —, *Quantum group structure associated to the quantum affine space*, Prépublication de IRMA, Strasbourg, Preprint 2001, No. **26**, Algebra Colloq. **11** (4) (2004), 483–492.
- [9] J.E. Humphreys, “*Introduction to Lie Algebras and Representation Theory*”, Graduate Texts in Math., vol. **9**, Springer-Verlag, New York/Berlin, 1978.
- [10] J.C. Jantzen, “*Lectures on Quantum Groups*”, Graduate Studies in Math., vol. **6**, Amer. Math. Soc., 1996.
- [11] C. Kassel, “*Quantum Groups*”, Graduate Texts in Math., vol. **155**, Springer-Verlag, New York/Berlin, 1995.
- [12] A. Klimyk and K. Schmüdgen, “*Quantum Groups and Their Representations*”, Springer-Verlag, Berlin/Heidelberg/New York, 1997.
- [13] Yunnan Li and Naihong Hu, *The Green rings of the 2-rank Taft algebra and its two relatives twisted*, J. Algebra **410** (2014), 1–35.
- [14] G. Lusztig, *Quantum deformations of certain simple modules over enveloping algebras*, Adv. in Math. **70** (1988), 237–249.
- [15] G. Lusztig, *On quantum groups*, J. Algebra **131** (1990), 466–475.
- [16] G. Lusztig, *Quantum groups at roots of 1*, Geom. Dedicata **35** (1990), 89–114.
- [17] G. Lusztig, “*Introduction to Quantum Groups*”, Progress in Mathematics, Vol. **110**, Birkhäuser, Basel, 1993.
- [18] S. Majid, *Double bosonisation of braided groups and the construction of $U_q(\mathfrak{g})$* , Math. Proc. Camb. Phil. Soc. **125** (1999) 151–192.
- [19] Yu. I. Manin, “*Quantum Groups and Non-commutative Geometry*”, Université de Montréal, 1988.
- [20] S. Montgomery and S.P. Smith, *Skew derivations and $U_q(\mathfrak{sl}_2)$* , Israel J. Math. **72**, (1/2) (1990), 158–166.
- [21] O. Ogievetsky, *Differential operators on quantum spaces for $GL_q(n)$ and $SO_q(n)$* , Lett. in Math. Phys. **24** (1992), 245–255.
- [22] D.E. Radford, “*Hopf Algebras*”, Series on Knots and Everything, vol. **49**, World Scientific, Singapore, 2012.
- [23] N.Yu. Reshetikhin, L.A. Takhtajan, and L.D. Faddeev, *Quantization of Lie groups and Lie algebras*, Algebra and Anal. **1** (1989), 178–206; Leningrad Math. J. **1** (1990), 193–225 [Engl. transl.]
- [24] P. Schauenburg, *Hopf modules and Yetter-Drinfeld modules*, J. Algebra **169** (1994), 874–890.
- [25] J. Wess and B. Zumino, *Covariant differential calculus on the quantum hyperplane*, Nucl. Phys. B (Proc. Supp.) **18** (1990), 302–312.
- [26] S.L. Woronowicz, *Differential calculus on compact matrix pseudogroups (quantum groups)*, Comm. Math. Phys. **122** (1989), 125–170.

DEPARTMENT OF MATHEMATICS, SHANGHAI KEY LABORATORY OF PURE MATHEMATICS AND
MATHEMATICAL PRACTICE, EAST CHINA NORMAL UNIVERSITY, MINHANG CAMPUS, DONG CHUAN
ROAD 500, SHANGHAI 200241, PR CHINA

E-mail address: `nhhu@math.ecnu.edu.cn`

DEPARTMENT OF MATHEMATICS, SHANGHAI UNIVERSITY, BAOSHAN CAMPUS, SHANGDA ROAD
99, SHANGHAI 200444, PR CHINA

E-mail address: `zhangjiao@i.shu.edu.cn`